Math 300 Spring-Summer 2018
Advanced Boundary Value Problems I

D'Alembert's Solution to the Wave Equation

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In this note we give a derivation of d'Alembert's solution to the one-dimensional wave equation and indicate how to use the superposition principle to decompose the original problem into two problems with different initial conditions and then combine the solutions to these two problems to get a solution to the original problem.

This method can then be used to find the general solution of the problem

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

with initial data

$$u(x,0) = f(x), \quad \frac{\partial u}{\partial t}(x,0) = g(x), \quad -\infty < x < \infty.$$

We start by using the change of variables

$$\alpha = x + ct \qquad \text{and} \qquad \beta = x - ct,$$

and using the chain rule to get

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial x} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial x} = \frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta},$$

and then replacing u by $\frac{\partial u}{\partial x}$, we get

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) = \frac{\partial}{\partial \alpha}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right) + \frac{\partial}{\partial \beta}\left(\frac{\partial u}{\partial \alpha} + \frac{\partial u}{\partial \beta}\right),$$

that is,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \alpha^2} + 2\frac{\partial^2 u}{\partial \alpha \partial \beta} + \frac{\partial^2 u}{\partial \beta^2}$$

Again, from the chain rule, we have

$$\frac{\partial u}{\partial t} = \frac{\partial u}{\partial \alpha}\frac{\partial \alpha}{\partial t} + \frac{\partial u}{\partial \beta}\frac{\partial \beta}{\partial t} = c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta},$$

and replacing u by $\dfrac{\partial u}{\partial t}$, we get

$$\frac{\partial^2 u}{\partial t^2} = \frac{\partial}{\partial t}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right) = c\frac{\partial}{\partial \alpha}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right) - c\frac{\partial}{\partial \beta}\left(c\frac{\partial u}{\partial \alpha} - c\frac{\partial u}{\partial \beta}\right),$$

that is,

$$\frac{\partial^2 u}{\partial t^2} = c^2\frac{\partial^2 u}{\partial \alpha^2} - 2c^2\frac{\partial^2 u}{\partial \alpha \partial \beta} + c^2\frac{\partial^2 u}{\partial \beta^2}.$$

Substituting these expressions into the wave equation, we obtain

$$\frac{\partial^2 u}{\partial\alpha\partial\beta} = 0.$$

This equation says that $\dfrac{\partial u}{\partial\beta}$ doesn't depend on α, and therefore

$$\frac{\partial u}{\partial\beta} = \phi(\beta),$$

where ϕ is an arbitrary differentiable function.

Integrating this equation with respect to β, holding α fixed, we get

$$u = \int \frac{\partial u}{\partial\beta}\, d\beta + F(\alpha) = \int \phi(\beta)\, d\beta + F(\alpha) = F(\alpha) + G(\beta),$$

where F is an arbitrary differentiable function and G is an antiderivative of ϕ.

Finally, using the fact that $\alpha = x+ct$ and $\beta = x-ct$, we get **d'Alembert's solution** to the one-dimensional wave equation:

$$u(x,t) = F(x+ct) + G(x-ct),$$

where F and G are arbitrary differentiable functions.

Now, in order to solve the original problem, we solve the following initial value – boundary value problems, and use the superposition principle to combine them to get a solution:

$$\begin{aligned} \frac{\partial^2 v}{\partial t^2} &= c^2 \frac{\partial^2 v}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geq 0, \\ v(x,0) &= f(x), \quad -\infty < x < \infty \\ \frac{\partial v}{\partial t}(x,0) &= 0 \quad -\infty < x < \infty, \end{aligned} \tag{1}$$

and

$$\begin{aligned} \frac{\partial^2 w}{\partial t^2} &= c^2 \frac{\partial^2 w}{\partial x^2}, \quad -\infty < x < \infty, \quad t \geq 0, \\ w(x,0) &= 0, \quad -\infty < x < \infty \\ \frac{\partial w}{\partial t}(x,0) &= g(x) \quad -\infty < x < \infty, \end{aligned} \tag{2}$$

the solution to the original problem is then $u = v + w$.

For problem (1), we use the initial conditions to write

$$v(x,0) = f(x) = F(x) + G(x),$$

so that

$$F(x) + G(x) = f(x),$$

and

$$\frac{\partial v}{\partial t} = 0 = cF'(x) - cG'(x),$$

so that

$$F(x) - G(x) = C,$$

where C is an arbitrary constant.

Therefore,

$$2F(x) = f(x) + C \qquad \text{and} \qquad 2G(x) = f(x) - C,$$

and the solution to the first problem is

$$v(x,t) = F(x+ct) + G(x-ct) = \frac{1}{2}\left[f(x+ct) + f((x-ct)\right].$$

For problem (2), we use the initial conditions to write

$$w(x,0) = 0 = F(x) + G(x),$$

so that $G(x) = -F(x)$, and

$$\frac{\partial w}{\partial t}(x,0) = g(x) = cF'(x) - cG'(x),$$

so that $cF'(x) - cG'(x) = 2cF'(x) = g(x)$, and integrating we have

$$2cF(x) = \int_0^x g(s)\,ds + 2cC,$$

where C is an arbitrary constant. Therefore,

$$F(x) = \frac{1}{2c}\int_0^x g(s)\,ds + C \qquad \text{and} \qquad G(x) = -\frac{1}{2c}\int_0^x g(s)\,ds - C$$

and the solution to the second problem is

$$w(x,t) = \frac{1}{2c}\left[\int_0^{x+ct} g(s)\,ds - \int_0^{x-ct} g(s)\,ds\right] = \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds.$$

The solution to the original initial value – boundary value problem is then

$$u(x,t) = v(x,t) + w(x,t) = \frac{1}{2}\left[f(x+ct) + f(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} g(s)\,ds$$

for $-\infty < x < \infty,\ t \geq 0$.