Math 300 Spring-Summer 2018
Advanced Boundary Value Problems I

General Solution to Laplace's Equation

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General Solution to Laplace's Equation

Given a linear equation with constant coeficients such as

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} = 0, \tag{$*$}$$

we can always find the general solution using the same technique we will use to find d'Alembert's general solution to the wave equation.

Consider the coordinate transformation

$$\begin{aligned} r &= ax + by \\ s &= cx + dy, \end{aligned}$$

where

$$\det\left(\frac{\partial(r,s)}{\partial(x,y)}\right) = ad - bc \neq 0.$$

From the chain rule, we have

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial x} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial x} = a\frac{\partial u}{\partial r} + c\frac{\partial u}{\partial s}$$

and

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r}\frac{\partial r}{\partial y} + \frac{\partial u}{\partial s}\frac{\partial s}{\partial y} = b\frac{\partial u}{\partial r} + d\frac{\partial u}{\partial s}.$$

Also,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial r^2}\left(\frac{\partial r}{\partial x}\right)^2 + 2\frac{\partial^2 u}{\partial r \partial s}\left(\frac{\partial r}{\partial x}\right)\left(\frac{\partial s}{\partial x}\right) + \frac{\partial^2 u}{\partial s^2}\left(\frac{\partial s}{\partial x}\right)^2,$$

and similarly,

$$\begin{aligned} \frac{\partial^2 u}{\partial x^2} &= a^2\frac{\partial^2 u}{\partial r^2} + 2ac\frac{\partial^2 u}{\partial r \partial s} + c^2\frac{\partial^2 u}{\partial s^2}, \\ \frac{\partial^2 u}{\partial x \partial y} &= ab\frac{\partial^2 u}{\partial r^2} + (ad+bc)\frac{\partial^2 u}{\partial r \partial s} + cd\frac{\partial^2 u}{\partial s^2}, \\ \frac{\partial^2 u}{\partial y^2} &= b^2\frac{\partial^2 u}{\partial r^2} + 2bd\frac{\partial^2 u}{\partial r \partial s} + d^2\frac{\partial^2 u}{\partial s^2}. \end{aligned}$$

Substituting these expressions into the partial differential equation $(*)$, we have

$$(Aa^2 + Bab + Cb^2)\frac{\partial^2 u}{\partial r^2} + (Ac^2 + Bcd + Cd^2)\frac{\partial^2 u}{\partial s^2} + (2Aac + B(ad + bc) + 2Cbd)\frac{\partial^2 u}{\partial r \partial s} = 0. \qquad (**)$$

By suitable choices of a, b, c, and d, we can make the first two terms in $(**)$ vanish. If $A \neq 0$, then among other possible choices we can put $b = d = 1$, and choose a and c to be the roots m_1 and m_2 of the quadratic equation

$$Am^2 + Bm + C = 0.$$

If we choose

$$a = m_1 = \frac{-B + \sqrt{B^2 - 4AC}}{2A} \qquad \text{and} \qquad c = m_2 = \frac{-B - \sqrt{B^2 - 4AC}}{2A},$$

then $(**)$ becomes

$$\left[2Am_1m_2 + B(m_1 + m_2) + 2C\right] \frac{\partial^2 u}{\partial r \partial s} = 0,$$

and since

$$m_1 + m_2 = -\frac{B}{A} \qquad \text{and} \qquad m_1 m_2 = \frac{C}{A},$$

this becomes

$$\frac{1}{A}\left(B^2 - 4AC\right)\frac{\partial^2 u}{\partial r \partial s} = 0. \qquad (***)$$

Both the elliptic and hyperbolic cases require that $B^2 - 4AC \neq 0$, and therefore $(***)$ reduces to

$$\frac{\partial^2 u}{\partial r \partial s} = 0,$$

with general solution

$$u(r, s) = F(r) + G(s),$$

where F and G are arbitrary twice differentiable functions.

Since

$$r = m_1 x + y$$
$$s = m_2 x + y,$$

we can write the general solution as

$$u(x, y) = F(m_1 x + y) + G(m_2 x + y),$$

where m_1 and m_2 are as given above.

The nature of the solution depends on whether the equation is elliptic, hyperbolic, or parabolic, in fact, we have the following results:

(i) If $B^2 - 4AC > 0$, then the equation is hyperbolic, and the roots m_1 and m_2 are real and distinct.

(ii) If $B^2 - 4AC < 0$, then the equation is elliptic, and the roots are complex conjugates of each other, so $m_2 = \overline{m}_1$.

(iii) If $B^2 - 4AC = 0$, then the equation is parabolic, and the roots are real and equal, so $m_1 = m_2 = m$. In this case we choose the linear transformation

$$r = mx + y$$
$$s = x,$$

then $a = m$, $b = 1$, $c = 1$, and $d = 0$, and $(**)$ becomes

$$\frac{\partial^2 u}{\partial s^2} = 0,$$

with general solution

$$u(r, s) = F(r) + sG(r)$$

(since $A \neq 0$). In terms of x and y, the general solution is

$$u(x, y) = F(mx + y) + xG(mx + y).$$

Note: If $A = 0$, the above results are still valid, provided we interchange the roles of x and y and the roles of A and C.

Example. (*Potential Equation*)

For Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0,$$

the discriminant is $B^2 - 4AC = -4 < 0$, and the equation is elliptic.

The roots of the quadratic equation $m^2 + 1 = 0$ are $m_1 = i$, and $m_2 = -i$, and the general solution to Laplace's equation is

$$u(x, y) = F(y + ix) + G(y - ix),$$

where F and G are arbitrary twice continuously differentiable functions.