Math 300 Spring-Summer 2018

Advanced Boundary Value Problems I

Interior Dirichlet Problem for Laplace's Equation in a Disk

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Example (Interior Dirichlet Problem for Laplace's Equation in a Disk) We want to solve the Dirichlet problem for Laplace's equation

$$\Delta u = 0$$

in the disk $D(a) = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 \leq a^2\}$. Here the appropriate coordinate system consists of plane polar coordinates r and θ, where $x = r\cos\theta$ and $y = r\sin\theta$. The disk above can then be described as $D(a) = \{(r, \theta) \mid 0 \leq r \leq a,\ -\pi \leq \theta \leq \pi\}$.

A formal statement of the problem is given below:

(i) The function $u(r, \theta)$ must satisfy *Laplace's equation in polar coordinates* r, θ, that is,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \qquad (*)$$

for $(r, \theta) \in D(a)$.

(ii) In order to ensure that the solution is single-valued, $u(r, \theta)$ must satisfy *periodicity conditions* at $\theta = \pm\pi$, that is,

$$u(r, -\pi) = u(r, \pi)$$

$$\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$

for $0 \leq r \leq a$.

(iii) In order to ensure that the solution is continuous, $u(r, \theta)$ must satisfy *boundedness conditions* at $r = 0$, that is,

$$\lim_{r \to 0^+} u(r, \theta) = u(0, \theta) \quad \text{(finite)}$$

for $-\pi \leq \theta \leq \pi$.

(iv) Finally, the solution must satisfy the *boundary condition* at $r = a$, that is,

$$u(a, \theta) = f(\theta)$$

for $-\pi \leq \theta \leq \pi$.

The interior Dirichlet problem for Laplace's equation on the disk $D(a)$ consists of (i), (ii), (iii), (iv), and this problem models, among other things, the steady-state temperature distribution of a circular plate with top and bottom perfectly insulated, and boundaries held at the temperatures given.

We look for a *separable solution*, that is, a solution of the form

$$u(r,\theta) = R(r)\Theta(\theta),$$

and substituting this into Laplace's equation $(*)$, we obtain

$$\frac{1}{r}\frac{d}{dr}\left(r\frac{dR}{dr}\right)\cdot\Theta + R\cdot\frac{1}{r^2}\frac{d^2\Theta}{d\theta^2} = 0,$$

that is,

$$r^2\,R''\cdot\Theta + r\,R'\cdot\Theta + R\cdot\Theta'' = 0.$$

Separating variable, we have

$$\frac{\Theta''}{\Theta} = -\frac{r^2R'' + r\,R'}{R} = -\lambda$$

where λ is the separation constant, and we have two ordinary differential equations:

- The *angle problem*:
$$\Theta'' + \lambda\Theta = 0$$
- The *radius problem*:
$$r^2R'' + r\,R' - \lambda R = 0$$

Note that the periodicity conditions (ii) imply that

$$R(r)\Theta(-\pi) = R(r)\Theta(\pi) \qquad \text{and} \qquad R(r)\Theta'(-\pi) = R(r)\Theta'(\pi)$$

for all $0 \le r \le a$, and in order to obtain a nontrivial solution, we must have

$$\Theta(-\pi) = \Theta(\pi)$$
$$\Theta'(-\pi) = \Theta'(\pi).$$

Therefore, Θ problem

$$\Theta'' + \lambda\Theta = 0$$
$$\Theta(-\pi) = \Theta(\pi)$$
$$\Theta'(-\pi) = \Theta'(\pi),$$

with eigenvalues and corresponding eigenfunctions given by **Check This!**

$$\lambda_0 = 0, \qquad \Theta_0(\theta) = a_0, \quad n = 0$$
$$\lambda_n = n^2 \qquad \Theta_n(\theta) = a_n\cos nx + b_n\sin nx, \quad n \ge 1.$$

The corresponding problem for R_n

$$r^2R'' + r\,R' - \lambda R = 0$$

is a Cauchy-Euler equation, and we assume a solution of the form $R(r) = r^s$, so that $R'(r) = sr^{s-1}$ and $R''(r) = s(s-1)r^{s-2}$, and substituting this into the equation we have

$$s(s-1)r^s + sr^s - \lambda r^s = 0,$$

and assuming $r \ne 0$, we get the characteristic equation

$$s(s-1) + s - \lambda = 0$$

that is, $s^2 = \lambda$, and $s = \pm\sqrt{\lambda}$.

Now we have to consider two cases:

(a) If $n = 0$, then $\lambda_n = 0$, and $s = 0$ is a double root of the characteristic equation, and one solution to the Euler equation is $R(r) = c_1$, that is, a constant solution. In order to find a second linearly independent solution, we consider the original differential equation for $\lambda = 0$,

$$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) = 0,$$

integrating,

$$r\frac{dR}{dr} = c_2,$$

so that

$$\frac{dR}{dr} = \frac{c_2}{r}$$

and a second independent solution is

$$R(r) = c_2 \log r.$$

The general solution to the radius equation for $\lambda_0 = 0$ is then

$$R_0(r) = c_1 + c_2 \log r$$

for $0 < r \le a$.

(b) If $n > 0$, then $\lambda_n^2 = n$, and $s = \pm n$, and the general solution to the radius equation for $\lambda_n = n^2$ is

$$R_n(r) = c_3 r^n + c_4 r^{-n}$$

for $0 < r \le a$.

From the boundedness condition (iii), we need

$$|u(r, \theta)| < \infty$$

as $r \to 0^+$, so we must have $c_2 = 0$ and $c_4 = 0$, and so

$$R_n(r) = r^n$$

for $n \ge 0$.

Using the superposition principle, we write

$$u(r, \theta) = \sum_{n=0}^{\infty} r^n \left(a_n \cos n\theta + b_n \sin n\theta\right) \qquad (**)$$

and determine the constants from the boundary condition (iv), so that

$$f(\varphi) = u(a, \varphi) = \sum_{n=0}^{\infty} a^n \left(a_n \cos n\varphi + b_n \sin n\varphi\right)$$

where

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi, \qquad a_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\varphi)\cos n\varphi\,d\varphi, \qquad b_n = \frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\varphi)\sin n\varphi\,d\varphi,$$

for $n \ge 1$.

Substituting these values of a_n and b_n into $(**)$, we have

$$\begin{aligned}
u(r,\theta) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\,d\varphi + \sum_{n=1}^{\infty}\frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\varphi) r^n \cos n\theta \cos n\varphi\,d\varphi + \sum_{n=1}^{\infty}\frac{1}{\pi a^n}\int_{-\pi}^{\pi} f(\varphi) r^n \sin n\theta \sin n\varphi\,d\varphi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left(1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\left(\cos n\theta\cos n\varphi + \sin n\theta \sin n\varphi\right)\right) d\varphi \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left(1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\cos n(\theta-\varphi)\right) d\varphi,
\end{aligned}$$

and

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left(1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\cos n(\theta-\varphi)\right) d\varphi \qquad (***)$$

for $0 \le r \le a,\ -\pi \le \theta \le \theta$.

We can easily sum the series inside the integral by noting that

$$2\cos n\theta\cos\theta = \cos(n+1)\theta + \cos(n-1)\theta,$$

and if $|b| < 1$, then

$$\begin{aligned}
2\sum_{n=1}^{\infty} b^n \cos n\theta\cos\theta &= \frac{1}{b}\sum_{n=1}^{\infty} b^{n+1}\cos(n+1)\theta + b\sum_{n=1}^{\infty} b^{n-1}\cos(n-1)\theta \\
&= \frac{1}{b}\left[\sum_{n=1}^{\infty} b^n\cos n\theta - b\cos\theta\right] + b\left[\sum_{n=1}^{\infty} b^n \cos n\theta + 1\right] \\
&= \left(\frac{1}{b} + b\right)\sum_{n=1}^{\infty} b^n\cos n\theta + b - \cos\theta,
\end{aligned}$$

so that

$$\left[1 - 2b\cos\theta + b^2\right]\sum_{n=1}^{\infty} b^n\cos n\theta = b\cos\theta - b^2,$$

and

$$\sum_{n=1}^{\infty} b^n \cos n\theta = \frac{b\cos\theta - b^2}{1 - 2b\cos\theta + b^2}.$$

Replacing b by $\dfrac{r}{a}$ and θ by $\theta - \varphi$ under the integral sign in $(***)$, we have

$$u(r,\theta) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\varphi)\left(1 + 2\frac{ar\cos(\theta-\varphi) - r^2}{a^2 - 2ar\cos(\theta-\varphi) + r^2}\right) d\varphi$$

and

$$u(r,\theta) = \frac{a^2 - r^2}{2\pi}\int_{-\pi}^{\pi}\frac{f(\varphi)}{a^2 - 2ar\cos(\theta-\varphi) + r^2}\,d\varphi \qquad (+)$$

for $0 \le r \le a,\ -\pi \le \theta \le \pi$. This is called **Poisson's integral formula for the disk** $D(a)$, and gives the unique solution to the interior Dirichlet problem for Laplace's equation on the disk.

Exercise. (Exterior Dirichlet Problem for a Disk) Find a solution to Laplace's equation $\Delta u = 0$ in the exterior of the disk $D(a)$, that is, find a bounded solution to the problem

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

$$u(r, -\pi) = u(r, \pi)$$

$$\frac{\partial u}{\partial \theta}(r, -\pi) = \frac{\partial u}{\partial \theta}(r, \pi)$$

$$u(a, \theta) = f(\theta)$$

for $a \le r < \infty, \ -\pi \le \theta \le \pi$.