<u>Example 2</u> Solve the homogeneous Neumann problem for the heat equation:

$$\left.\begin{aligned}
&\frac{\partial u}{\partial t} = \sqrt{5}\,\frac{\partial^2 u}{\partial x^2} \qquad \text{on} \quad 0 < x < 2 \\
&\frac{\partial u}{\partial x}(0,t) = 0 \quad , \quad \frac{\partial u}{\partial x}(2,t) = 0 \\
&u(x,0) = 15\cos\left(\tfrac{3}{2}\pi x\right) + 16\cos\left(\tfrac{171}{2}\pi x\right)
\end{aligned}\right\}$$

Separation of variables:

$$u(x,t) = G(t)\,\varphi(x)$$

$$\frac{\partial u}{\partial t} = G'\varphi \quad , \quad \frac{\partial^2 u}{\partial x^2} = G\varphi''$$

$$\Rightarrow \quad G'\varphi = \sqrt{5}\, G\varphi''$$

$$\Rightarrow \quad \frac{G'(t)}{\sqrt{5}\, G(t)} = \frac{\varphi''(x)}{\varphi(x)} = -\lambda \qquad \text{some constant.}$$

time equation $\quad G'(t) = -\sqrt{5}\,\lambda\, G(t)$

space equation $\quad \varphi''(x) = -\lambda\, \varphi(x)$

boundary conditions:

$$G(t)\,\varphi'(0) = 0 = G(t)\,\varphi'(2)$$

If $G(t) = 0 \Rightarrow$ trivial solution.

Study $\varphi'(0) = 0, \quad \varphi'(L) = 0.$

<u>1. The spatial problem</u>

$$\left.\begin{array}{l} \varphi''(x) = -\lambda\,\varphi(x) \\ \varphi'(0) = 0, \ \varphi'(L) = 0 \end{array}\right\} \text{ on } [0,2].$$

<u>Case 1</u>: $\lambda < 0$. $\quad \varphi(x) = c_1 e^{\sqrt{-\lambda}\,x} + c_2 e^{-\sqrt{-\lambda}\,x}$

$$\varphi(x) = \sqrt{-\lambda}\, c_1 e^{\sqrt{-\lambda}\,x} - \sqrt{-\lambda}\, c_2 e^{\sqrt{-\lambda}\,x}$$

$$0 = \varphi'(0) = \sqrt{-\lambda}\,(c_1 - c_2) \Rightarrow \quad c_1 = c_2$$

$$0 = \varphi'(2) = \sqrt{-\lambda}\, c_1 \left(e^{\sqrt{-\lambda}\, 2} - e^{-2\sqrt{-\lambda}} \right) \Rightarrow c_1 = 0$$

$$\Rightarrow c_2 = 0$$

$\Rightarrow$ trivial solution.

<u>Case 2</u> $\lambda = 0$: $\varphi(x) = c_1 x + c_2$, $\varphi'(x) = c_1$

$$0 = \varphi'(0) = c_1 \Rightarrow c_1 = 0.$$

$$\Rightarrow \varphi(x) = c_2$$

<u>Case 3</u> $\lambda > 0$: $\varphi(x) = c_1 \cos(\sqrt{\lambda}\, x) + c_2 \sin(\sqrt{\lambda}\, x)$

$$\varphi'(x) = -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda}\, x) + \sqrt{\lambda}\, c_2 \cos(\sqrt{\lambda}\, x)$$

$$0 = \varphi'(0) = \sqrt{\lambda}\, c_2 \Rightarrow c_2 = 0$$

$$0 = \varphi'(2) = -\sqrt{\lambda}\, c_1 \sin(\sqrt{\lambda}\, 2)$$

$\Rightarrow$ either $c_1 = 0 \Rightarrow$ trivial solution

or $\sqrt{\lambda}\, 2 = n\pi$, $n \in \mathbb{N} \cup \{0\}$

$\Rightarrow \lambda = \left(\frac{n\pi}{2}\right)^2$ (eigenvalues)

Corresponding eigenfunctions: $\cos\left(\frac{n\pi}{2} x\right)$, $n \in \mathbb{N}_0$

2. The time problem: For each $n \in \mathbb{N}_0$ we solve the time-problem

$$G_n'(t) = -\left(\frac{n\pi}{2}\right)^2 \sqrt{5}\, G(t)$$

$$\Rightarrow \quad G_n(t) = c_n\, e^{-\left(\frac{n\pi}{2}\right)^2 \sqrt{5}\, t}$$

Via superposition we find the general solution:

$$u(x,t) = \sum_{n=0}^{\infty} c_n\, e^{-\left(\frac{n\pi}{2}\right)^2 \sqrt{5}\, t} \cos\left(\frac{n\pi}{2} x\right) \qquad (*)$$

3. Initial condition:

$$u(x,0) = \sum_{n=0}^{\infty} c_n \cos\left(\frac{n\pi}{2} x\right)$$

the Fourier-cosine-series of the initial condition.

In this example:

$$u(x,0) = 15 \cos\left(\frac{3}{2}\pi x\right) + 16 \cos\left(\frac{171}{2}\pi x\right)$$

hence $c_3 = 15$, $c_{171} = 16$

all other $c_j = 0 \quad j \neq 3,\ j \neq 171.$

Then we find from (*)

$$u(x,t) = 15\, e^{-\sqrt{5}\,\frac{9\pi^2}{4}t} \cos\left(\tfrac{3\pi}{2}x\right)$$

$$+ 16\, e^{-\sqrt{5}\,\frac{29\,241}{4}\pi^2 t} \cos\left(\tfrac{171\pi}{2}x\right)$$

<u>Summary</u> The heat-equation Neumann problem

$$\left.\begin{aligned} &\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2}, \quad 0 < x < L \\ &\frac{\partial u}{\partial x}(0,t) = \frac{\partial u}{\partial x}(L,t) = 0 \\ &u(x,0) = f(x) \end{aligned}\right\}$$

Is solved by $u(x,t) = \sum_{n=0}^{\infty} c_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L}x\right)$

Where $c_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi}{2}x\right)dx$

are the Fourier-cosine coefficients of $f(x)$.

$$c_0 = \frac{1}{L}\int_0^L f(x)\,dx$$

The homogeneous Neumann problem of the heat equation ~~is solved~~

$$\begin{cases} u_t = k u_{xx} \quad \text{on } [0,L] \\ u_x(0,t) = 0, \quad u_x(L,t) = 0 \\ u(x,0) = f(x) \end{cases}$$

is solved by

$$u(x,t) = \sum_{n=0}^{\infty} C_n e^{-k\left(\frac{n\pi}{L}\right)^2 t} \cos\left(\frac{n\pi}{L} x\right)$$

$$C_n = \frac{2}{L} \int_0^L f(x) \cos\left(\frac{n\pi}{L} x\right) dx \qquad n > 0$$

$$C_0 = \frac{1}{L} \int_0^L f(x)\, dx$$

C_n are the coefficients of the Fourier-cosine series of $f(x)$.