(2.2) Linearity and Homogeneity

<u>Def</u>: A linear operator L is linear, if for two functions u, v and two constants α, β we always have

$$L[\alpha u + \beta v] = \alpha L[u] + \beta L[v].$$

<u>Example 1</u>: $L = A \in M_{2\times 2}$ matrix:

$$A\left(\alpha \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}\right) = \alpha A \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \beta A \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

<u>Example 2</u> derivation $L = \frac{\partial}{\partial t}$:

$$\frac{\partial}{\partial t}\left[\alpha u(t) + \beta v(t)\right] = \alpha \frac{\partial u}{\partial t} + \beta \frac{\partial v}{\partial t}$$

2^{nd} derivative $\frac{d^2}{dx^2}$:

$$\frac{d^2}{dx^2}\left[\alpha f(x) + \beta g(x)\right] = \alpha \frac{d^2}{dx^2} f + \beta \frac{d^2}{dx^2} g$$

<u>Example 3</u> Integration $L = \int_a^b dx$

$$L[\alpha u + \beta v] = \int_a^b \alpha u(x) + \beta v(x)\, dx$$

$$= \alpha \int_a^b u(x)\, dx + \beta \int_a^b v(x)\, dx$$

$$= \alpha L[u] + \beta L[v].$$

Linear or nonlinear ?

1) wave eq: $u_{tt} = c^2 u_{xx} + 3$

2) Fischer eq: $u_t = D u_{xx} + u(1-u)$

3) Korteweg-deVries eq:

$$u_t + \beta u u_x = \alpha u_{xxx}$$

4) Schrödinger eq

$$A_t + \omega A_x = i\gamma A_{xx}$$

5) Laplace eq $\Delta u = 0$

6) Porous medium eq.

$$u_t = \frac{\partial}{\partial x}\left(u \frac{\partial u}{\partial x}\right)$$

Def: A linear equation has the form $L[u] = f$, where L is a linear operator.

Example: heat equation $u_t = D u_{xx} + f(x)$

can be written as $L[u] = f$

with linear operator

$$L[u] = \frac{\partial}{\partial t} u - D \frac{\partial^2}{\partial x^2} u .$$

nonlinear: $\frac{\partial}{\partial t} u^2 = f$

Which of the examples on the transparency are linear?

1) Yes 4) Yes

2) No 5) Yes

3) No 6) No

Def: A ~~linear~~ equation is homogeneous if it has the trivial solution $u \equiv 0$.

1) No 4) Yes

2) Yes 5) Yes

3) Yes 6) Yes

Principle of Superposition

solution of the homogeneous equation:

$$L[\alpha u_1 + \beta u_2] = \alpha L[u_1] + \beta L[u_2] = 0.$$

Boundary conditions can be classified the same way

linear: $u(0,t) = f(t)$, $u(L,t) = g(t)$

$u_x(0,t) = \phi(t)$, $u_x(L,t) = \psi(t)$

homogeneous: $u(0,t) = 0$, $u(L,t) = 0$

etc.

nonlinear, non homogeneous:

$\tan(u(0,t)) = 3$, $\exp(u_x(L,t)) = 10$