(1.4) Equilibrium or Steady States

Example 1: Find the equilibrium temperature of a rod of length L with given temperature at the end as

$$u(0,t) = T_1 \qquad u(L,t) = T_2$$

Example 2 Find the equilibrium temp. of a rod of length L with perfectly insulated ends and initial temperature

$$u(x,0) = 3x^2$$

(1.4) Equilibrium Temperature

<u>Def</u>: A solution to a PDE that does <u>not</u> depend on time is called <u>steady state</u> or equilibrium.

<u>Example 1</u>: heat equation

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} \quad \text{on } [0, L]$$

$$u(0,t) = T_1, \quad u(L,t) = T_2$$

(Problem: not enough conditions!)

Maybe that's OK for equilibrium? Let's try:

$$u \text{ equilibrium} \Rightarrow u(x, \cancel{t})$$

$$\Rightarrow \frac{\partial}{\partial t} u(x,t) = 0$$

$$\cancel{\Rightarrow \frac{\partial^2 u}{\partial x^2}(x,t) = 0}$$

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} = 0$$

$$\Rightarrow \frac{\partial}{\partial x} u(x) = C_1$$

$$\Rightarrow u(x) = C_1 x + C_2$$

Find C_1 and C_2 from the boundary conditions:

$$T_1 = u(0) = C_2$$

$$T_2 = u(L) = C_1 L + C_2 = C_1 L + T_1$$

$$\Rightarrow \quad C_1 = \frac{T_2 - T_1}{L}$$

Hence the equilibrium temperature is

$$\boxed{u(x) = \frac{T_2 - T_1}{L} x + T_1}$$

Example 2

$$\left.\begin{aligned} &\frac{\partial}{\partial t} u = k \frac{\partial^2 u}{\partial x^2} \quad \text{on } [0,L] \\ &\frac{\partial}{\partial x} u(0,t) = 0 \ , \ \frac{\partial}{\partial x} u(L,t) = 0 \\ &u(x,0) = 3x^2 \qquad x \in [0,L] \end{aligned}\right\}$$

Equilibrium: $k \frac{\partial^2}{\partial x^2} u(x) = 0$

$$\Rightarrow \quad u''(x) = 0$$

$$\Rightarrow \quad u'(x) = C_1$$

$$u(x) = C_1 x + C_2 \ .$$

boundary conditions: $u'(x) = C_1$

$$u'(0) = 0 \Rightarrow C_1 = 0$$

$$u'(L) = 0 \Rightarrow C_1 = 0$$

$$\Rightarrow \boxed{u(x) = C_2 \, !}$$ But what is C_2 ?

Physically: both ends are insulated, i.e. the total heat energy should be constant.

$$E = \text{Const} = \int_0^L c\rho\, u(x)\, dx = \int_0^L c\rho\, u(x,0)\, dx$$

initial condition !

$$= c\rho \int_0^L 3x^2 dx = c\rho L^3$$

$$\Rightarrow \int_0^L u(x)\,dx = L^3 \qquad u(x) = C_2$$

$$C_2 L = L^3$$

$$C_2 = L^2$$

Thus the unique solution is

$$\boxed{u(x) = L^2}$$

(1.5) Heat equation in 2-D and 3-D

$$\frac{\partial u(x,y,t)}{\partial t} = D\left(\frac{\partial^2}{\partial x^2} u(x,y,t) + \frac{\partial^2}{\partial y^2} u(x,y,t)\right)$$

$$\frac{\partial u(x,y,z,t)}{\partial t} = D\left(\frac{\partial^2}{\partial x^2} u(x,y,z,t) + \frac{\partial^2}{\partial y^2} u(x,y,z,t) + \frac{\partial^2}{\partial z^2} u(x,y,z,t)\right)$$

The differential operator of the right hand side is called Laplace-operator: Δ

$$\Delta u(x,y,t) = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u$$

$$\Delta u(x,y,z,t) = \frac{\partial^2}{\partial x^2} u + \frac{\partial^2}{\partial y^2} u + \frac{\partial^2}{\partial z^2} u$$

$$\Delta u(x,t) = \frac{\partial^2}{\partial x^2} u$$

heat equation ~~also~~ $\boxed{u_t = D\,\Delta u}$

Nabla $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$

$f(x,y,z)$: $\nabla f(x,y,z) = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right)$

gradient of f.

$$\nabla^2 = \Delta.$$

We can express Δ in polar, cylindrical or spherical coordinates:

<u>Example</u>: Write the 2-D heat equation in polar coordinates:

$$x = r\cos\theta, \quad y = r\sin\theta$$

$$u(x,y,t) = u(x(r,\theta), y(r,\theta), t)$$

for later use: $\frac{\partial x}{\partial r} = \cos\theta \quad \frac{\partial x}{\partial \theta} = -r\sin\theta$

$$\frac{\partial y}{\partial r} = \sin\theta \quad \frac{\partial y}{\partial \theta} = r\cos\theta$$

$$\frac{\partial}{\partial r} u(t, x(r,\theta), y(r,\theta)) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r} \quad \text{chain rule}$$

$$= \cos\theta \frac{\partial u}{\partial x} + \sin\theta \frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial \theta} u(x(r,\theta), y(r,\theta), t) = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta} = -r\sin\theta \frac{\partial u}{\partial x} + r\cos\theta \frac{\partial u}{\partial y}$$

$$\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}u\right)=\frac{\partial}{\partial r}\left(r\cos\theta\frac{\partial u}{\partial x}+r\sin\theta\frac{\partial u}{\partial y}\right)$$

$$=\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}+r\cos\theta\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial r}\right)$$

$$+r\sin\theta\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial r}\right)$$

$$=\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}+r\cos\theta\frac{\partial}{\partial x}\left(\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}\right)$$

$$+r\sin\theta\frac{\partial}{\partial y}\left(\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}\right)$$

$$=\cos\theta\frac{\partial u}{\partial x}+\sin\theta\frac{\partial u}{\partial y}+r\cos^2\theta\frac{\partial^2}{\partial x^2}u+2r\cos\theta\sin\theta\frac{\partial^2}{\partial x\partial y}u$$

$$+r\sin^2\theta\frac{\partial^2}{\partial y^2}u$$

$$\frac{\partial}{\partial\theta}\left(\frac{\partial}{\partial\theta}u\right)=\frac{\partial}{\partial\theta}\left(-r\sin\theta\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y}\right)$$

$$=-r\cos\theta\frac{\partial u}{\partial x}-r\sin\theta\frac{\partial u}{\partial y}-r\sin\theta\frac{\partial}{\partial x}\left(-r\sin\theta\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y}\right)$$

$$+r\cos\theta\frac{\partial}{\partial y}\left(-r\sin\theta\frac{\partial u}{\partial x}+r\cos\theta\frac{\partial u}{\partial y}\right)$$

$$=-r\cos\theta\frac{\partial u}{\partial x}-r\sin\theta\frac{\partial u}{\partial y}+r^2\sin^2\theta\frac{\partial^2 u}{\partial x^2}-2r^2\cos\theta\sin\theta\frac{\partial^2}{\partial x\partial y}u$$

$$+r^2\cos^2\theta\frac{\partial^2}{\partial y^2}u$$

Then $\frac{1}{r}\frac{\partial}{\partial r}\left(r\cdot\frac{\partial}{\partial r}u\right)+\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}u=\frac{\partial^2}{\partial x^2}u+\frac{\partial^2}{\partial y^2}u$

heat equation in polar coordinates:

$$\frac{\partial}{\partial t} u(r, \theta, t) = \frac{1}{r} \frac{\partial}{\partial r}\left(r \cdot \frac{\partial}{\partial r} u\right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} u$$

Laplace operator in polar coordinates

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r}\left(r \cdot \frac{\partial}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2}$$

cylindrical (r, θ, z)

$$\Delta = \frac{1}{r} \frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2}$$

spherical (ρ, ϕ, θ)

$$x = \rho \sin\phi \cos\theta$$

$$y = \rho \sin\phi \sin\theta$$

$$z = \rho \cos\phi$$

$$\Delta = \frac{1}{\rho^2} \frac{\partial}{\partial \rho}\left(\rho^2 \frac{\partial}{\partial \rho}\right) + \frac{1}{\rho^2 \sin\phi} \frac{\partial}{\partial \phi}\left(\sin\phi \frac{\partial u}{\partial \phi}\right) + \frac{1}{\rho^2 \sin^2\phi} \frac{\partial^2 u}{\partial \theta^2}$$

Example: In 2-D, find Δf for $f(x,y) = x^2 + y^2$

cartesian: $$\Delta f = \frac{\partial^2}{\partial x^2} f + \frac{\partial^2}{\partial y^2} f = 2 + 2 = 4$$

polar: $\cancel{\Delta f}$ $f(r,\theta) = r^2$

$$\Delta f = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r} f\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} f$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(r(2r)\right)$$

$$= \frac{1}{r}\frac{\partial}{\partial r}\left(2r^2\right)$$

$$= 4$$