(1.1) What is a PDE?

advection equation

$$\frac{\partial}{\partial t} f(x,t) + c \frac{\partial}{\partial x} f(x,t) = 0$$

Wave equation

$$\frac{\partial^2}{\partial t^2} u(x,t) - \omega^2 \frac{\partial^2}{\partial x^2} u(x,t) = 0$$

heat equation

$$\frac{\partial}{\partial t} u(x,y,t) = D\left(\frac{\partial^2}{\partial x^2} u(x,y,t) + \frac{\partial^2}{\partial y^2} u(x,y,t) \right)$$

Laplace equation

2-D: $$\frac{\partial^2}{\partial x^2} w(x,y) + \frac{\partial^2}{\partial y^2} w(x,y) = 0$$

1-D: $$\frac{d^2}{dx^2} g(x) = 0$$

- Maxwell equations
- Schrödinger equations
- Navier-Stokes equations
- Population dynamics
- Physiology

The partial derivative

$D \subseteq \mathbb{R}^2, \quad f: D \to \mathbb{R}, \quad f(x,y)$

$$\frac{\partial f(x,y)}{\partial x} = \lim_{h \to 0} \frac{f(x+h,y) - f(x,y)}{h}$$

$$= f_x \qquad \text{index notation}$$

$$\frac{\partial f(x,y)}{\partial y} = \lim_{h \to 0} \frac{f(x,y+h) - f(x,y)}{h}$$

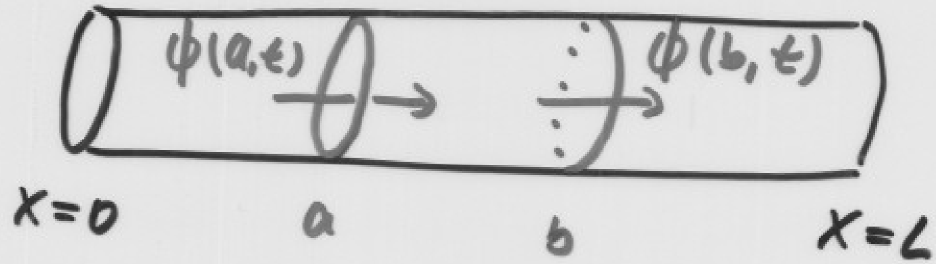
$$= f_y$$

Def: A partial differential equation (PDE) is an equation that involves partial derivatives

Goal
— formulation
— solution
— interpretation

(1.2) Derivation of the heat equation

rod:



thermal energy density $e(x,t)$

total energy in $[a,b]$: $\int_a^b e(x,t)\,dx$

Word equation:

rate of change of heat energy in $[a,b]$ = heat energy flowing across the boundary + heat sources or sinks inside $[a,b]$

heat flux $\phi(x,t)$

$$\boxed{\frac{d}{dt}\int_a^b e(x,t)\,dx = \phi(a,t) - \phi(b,t) + \int_a^b Q(x,t)\,dx}$$

heat sources or sinks $Q(x,t)$.

u

$$\Rightarrow \int_a^b \left(\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q \right) dx = 0$$

Since this is true for all $a < b$, we need to have

$$\frac{\partial e}{\partial t} + \frac{\partial \phi}{\partial x} - Q = 0$$

$$\boxed{\frac{\partial e}{\partial t} = -\frac{\partial \phi}{\partial x} + Q} \qquad ([illegible])$$

temperature $u(x,t)$

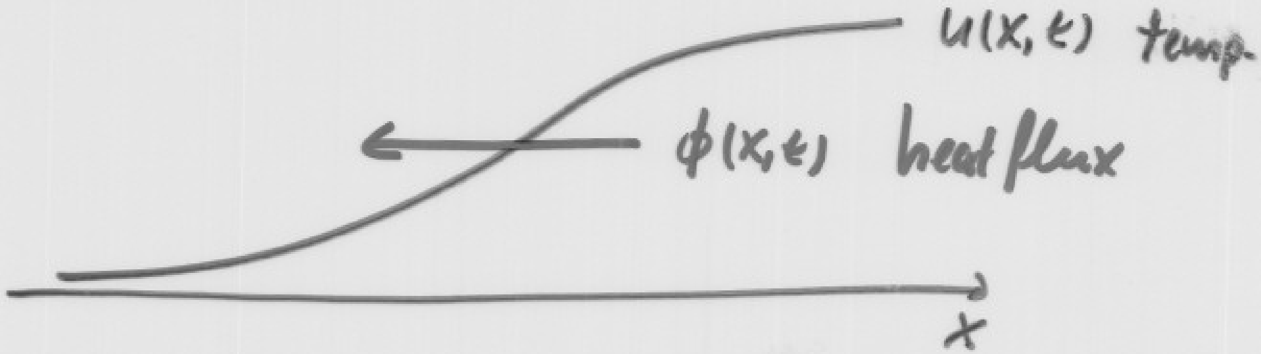
specific heat $c(x)$

mass density $\rho(x)$

thermal energy density

$$\boxed{e(x,t) = c(x)\,\rho(x)\,u(x,t)} \qquad (1.2.6)$$

<u>Fourier's law</u>



$$\boxed{\phi(x,t) = -K_0 \frac{\partial u(x,t)}{\partial x}} \qquad (1.2.8)$$

Fourier's law of heat conduction

model !

$$c(x) = \text{const}$$
$$\rho(x) = \text{const}$$
$$K_0 = \text{const}.$$

and obtain

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(\frac{K_0}{c\rho}\frac{\partial u}{\partial x}\right) + \frac{Q(x,t)}{c\rho}$$

introduce $k = \frac{K_0}{c\rho}, \quad q := \frac{Q}{c\rho}$

then we obtain the classical heat equation with source term:

$$\boxed{\frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} + q(x,t)}$$

Interpretations

$$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2} + q$$

1) heat equation: $u(x,t)$ temperature
   q : heat source
2) diffusion equation: $u(x,t)$ concentration of a chemical substance in solution
   q : chemical reactions

3) population dynamics : $u(x,t)$ population density (e.g. mountain pine beetle)
   q: reproduction and death

4) forest fires : $u(x,t)$ fire front
5) epidemic spread
6) cancer progression

etc. etc.

1.3 Side conditions

Hillen's rule of thumb:

Let $x, y, t, \ldots$ denote the independent variables.
The highest order partial derivative in each of the independent variables gives the number of side conditions needed.

Examples:

	# side conditions
heat equation	
$\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}$	$1+2=3$
advection eq:	
$\frac{\partial f}{\partial t} + c \frac{\partial f}{\partial x} = 0$	$1+1=2$
2-D wave eq	
$\frac{\partial^2 u}{\partial t^2} = c^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right)$	$2+2+2=6$

What are appropriate side conditions?

heat equation

$u(x,t)$?

(i) What is $u(x,t)$ at the beginning of our observation:

Initial condition $\boxed{u(x,0) = f(x)}$

(ii) What happens at the boundary points $x=0$ and $x=L$?

e.g. fixed temperature at $0\,°C$:

boundary conditions

$$\boxed{\begin{aligned} u(0,t) &= 0 \\ u(L,t) &= 0 \end{aligned}}$$

Boundary conditions

1) First kind
Dirichlet boundary conditions:
prescribed temperature:

$$u(0,t) = f_0(t), \quad u(L,t) = f_L(t)$$

homogeneous Dirichlet b.c.:

$$u(0,t) = 0, \quad u(L,t) = 0$$

2) Second kind
Neumann boundary conditions:
prescribed heat flux:

$$\frac{\partial}{\partial x} u(0,t) = \phi_0(t), \quad \frac{\partial}{\partial x} u(L,t) = \phi_L(t)$$

homogeneous Neumann b.c. (perfectly insulated)

$$u_x(0,t) = 0, \quad u_x(L,t) = 0$$

3) Third kind
Robin boundary conditions
combination of heat flux and temperature,
e.g. Newton's law of cooling:

$$u_x(0,t) = -h_0\left(u(0,t) - f_0(t)\right)$$

$$u_x(L,t) = -h_L\left(u(L,t) - f_L(t)\right)$$