MATH 243 Winter 2008
Geometry II: Transformation Geometry
Solutions to Problem Set 5
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Question 1. *Thomsen's Relation* Prove that for any lines a, b, c :

$$\sigma_c\,\sigma_a\,\sigma_b\,\sigma_c\,\sigma_a\,\sigma_b\,\sigma_a\,\sigma_b\,\sigma_c\,\sigma_a\,\sigma_b\,\sigma_c\,\sigma_b\,\sigma_a\,\sigma_c\,\sigma_b\,\sigma_a\,\sigma_b\,\sigma_a\,\sigma_c\,\sigma_b\,\sigma_a = \iota.$$

SOLUTION: First we note that $(\sigma_c\sigma_a\sigma_b)^2$ is a translation.

If a, b, c are concurrent or parallel, then $\sigma_c\sigma_a\sigma_b = \sigma_m$ is a reflection in a line m, and

$$(\sigma_c\sigma_a\sigma_b)^2 = \sigma_m^2 = \iota = \tau_{\overrightarrow{0}}.$$

If a, b, c are neither concurrent nor parallel, then $\sigma_c\sigma_a\sigma_b$ is a glide reflection, say

$$\sigma_c\sigma_a\sigma_b = \tau_{\overrightarrow{u}}\sigma_\ell = \sigma_\ell\tau_{\overrightarrow{u}},$$

where $\overrightarrow{u}$ is parallel to ℓ, so that

$$(\sigma_c\sigma_a\sigma_b)^2 = \tau_{\overrightarrow{u}}\sigma_\ell\sigma_\ell\tau_{\overrightarrow{u}} = \tau_{\overrightarrow{u}}^2\sigma_\ell^2 = \tau_{\overrightarrow{u}}^2 = \tau_{\overrightarrow{2u}}.$$

We have

$$\begin{aligned}
&\sigma_c\,\sigma_a\,\sigma_b\,\sigma_c\,\sigma_a\,\sigma_b\,\sigma_a\,\sigma_b\,\sigma_c\,\sigma_a\,\sigma_b\,\sigma_c\,\sigma_b\,\sigma_a\,\sigma_c\,\sigma_b\,\sigma_a\,\sigma_b\,\sigma_a\,\sigma_c\,\sigma_b\,\sigma_a \\
&= \sigma_c\,\sigma_a\,\sigma_b\,\sigma_c\,\sigma_a\,\sigma_b\,\sigma_a\,\sigma_b\,\sigma_c\,\sigma_a\,\sigma_b\,\sigma_c\,\sigma_b\,\sigma_a\,\sigma_c\,\sigma_b\,\sigma_a\,\sigma_c\,\sigma_c\,\sigma_b\,\sigma_a\,\sigma_c\,\sigma_b\,\sigma_a \\
&= (\sigma_c\,\sigma_a\,\sigma_b)^2(\sigma_a\,\sigma_b\,\sigma_c)^2(\sigma_b\,\sigma_a\,\sigma_c)^2(\sigma_c\,\sigma_b\,\sigma_a)^2.
\end{aligned}$$

Since $(\sigma_c\,\sigma_b\,\sigma_a)^2$, $(\sigma_c\,\sigma_a\,\sigma_b)^2$, $(\sigma_b\,\sigma_a\,\sigma_c)^2$, and $(\sigma_a\,\sigma_b\,\sigma_c)^2$ are all translations, they commute.

Now note that

$$\begin{aligned}
(\sigma_c\,\sigma_b\,\sigma_a)^2(\sigma_a\,\sigma_b\,\sigma_c)^2 &= \sigma_c\,\sigma_b\,\sigma_a\,\sigma_c\,\sigma_b\,\sigma_a\,\sigma_a\,\sigma_b\,\sigma_c\,\sigma_a\,\sigma_b\,\sigma_c \\
&= \sigma_c\,\sigma_b\,\sigma_a\,\sigma_c\,\sigma_b\,\sigma_b\,\sigma_c\,\sigma_a\,\sigma_b\,\sigma_c \\
&= \sigma_c\,\sigma_b\,\sigma_a\,\sigma_c\,\sigma_c\,\sigma_a\,\sigma_b\,\sigma_c \\
&= \sigma_c\,\sigma_b\,\sigma_a\,\sigma_a\,\sigma_b\,\sigma_c \\
&= \sigma_c\,\sigma_b\,\sigma_b\,\sigma_c \\
&= \sigma_c\,\sigma_c \\
&= \iota.
\end{aligned}$$

Therefore,

$$(\sigma_c\,\sigma_b\,\sigma_a)^2 = \left((\sigma_a\,\sigma_b\,\sigma_c)^2\right)^{-1}$$

and

$$(\sigma_c\,\sigma_a\,\sigma_b)^2 = \left((\sigma_b\,\sigma_a\,\sigma_c)^2\right)^{-1},$$

so that

$$(\sigma_c\,\sigma_a\,\sigma_b)^2(\sigma_a\,\sigma_b\,\sigma_c)^2(\sigma_b\,\sigma_a\,\sigma_c)^2(\sigma_c\,\sigma_b\,\sigma_a)^2 = \iota,$$

and Thomsen's relation holds.

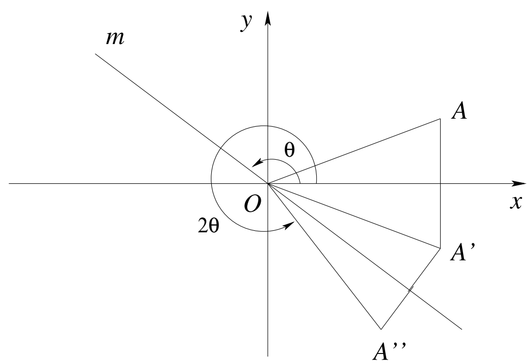
Question 2. If $x' = ax + by + c$ and $y' = bx - ay + d$ with $a^2 + b^2 = 1$ are the equations for an isometry α, show that α is a reflection if and only if

$$ac + bd + c = 0 \qquad \text{and} \qquad ad - bc - d = 0.$$

SOLUTION: First we show that if m is a line through the origin making a directed angle θ with the positive x-axis, and σ_x is a reflection in the x-axis, then

$$\rho_{O,2\theta} = \sigma_m \, \sigma_x,$$

therefore $\sigma_m = \rho_{O,2\theta} \, \sigma_x$.



Let A be an arbitrary point in the plane and let $A' = \sigma_x(A)$ and $A'' = \sigma_m(A')$, so that $A'' = \sigma_m\sigma_x(A)$. From the figure we see that $A''(A) = \rho_{O,2\theta}(A)$, and since A is arbitrary then

$$\rho_{O,2\theta} = \sigma_m \, \sigma_x.$$

Since

$$\sigma_x(x, y) = (x, -y) \qquad \text{and} \qquad \rho_{O,2\theta}(x, y) = (x \cos 2\theta - y \sin 2\theta, x \sin 2\theta + y \cos 2\theta),$$

then the equations of the reflection σ_m are given by

$$\begin{aligned} x' &= x \cos 2\theta + y \sin 2\theta \\ y' &= x \sin 2\theta - y \cos 2\theta \end{aligned}$$

By translating, rotating, and translating back, the equations of a reflection in a line ℓ passing through the point (h, k) and making a directed angle θ with the positive x-axis are given by

$$\begin{aligned} x' &= (x - h) \cos 2\theta + (y - k) \sin 2\theta + h \\ y' &= (x - h) \sin 2\theta - (y - k) \cos 2\theta + k \end{aligned}$$

Now, if α is an isometry with equations $x' = ax + by + c$ and $y' = bx - ay + d$ with $a^2 + b^2 = 1$, by letting $a = \cos 2\theta$ and $b = \sin 2\theta$, then from the above we see that α is a reflection if and only if

$$c = h - ah - kb \qquad \text{and} \qquad d = k - bh + ka,$$

and this is the case if and only if

$$ac + bd + c = 0 \qquad \text{and} \qquad ad - bc - d = 0.$$

Question 3. If $x' = \frac{3}{5}x + \frac{4}{5}y$ and $y' = \frac{4}{5}x - \frac{3}{5}y$ are the equations for σ_m, then find the line m.

SOLUTION: From the previous problem we see that the line m passes through the origin $(0,0)$, and

$$\cos 2\theta = \frac{3}{5}, \qquad \text{and} \qquad \sin 2\theta = \frac{4}{5},$$

so that

$$\cos 2\theta = \cos^2\theta - \sin^2\theta = \frac{3}{5} \qquad \text{and} \qquad \sin 2\theta = 2\sin\theta\cos\theta = \frac{4}{5}$$

implies

$$\cos\theta = \frac{2}{\sqrt{5}} \qquad \text{and} \sin\theta = \frac{1}{5}.$$

Therefore the slope of m is $\tan\theta = \dfrac{1}{2}$, so the equation of the line is $y = \dfrac{1}{2}x$.

Alternatively, we can look for fixed points of σ_m, solving

$$\begin{aligned} x = x' &= \frac{3}{5}x + \frac{4}{5}y \\ y = y' &= \frac{4}{5}x - \frac{3}{5}y, \end{aligned}$$

we find infinitely many solutions: $y = \dfrac{1}{2}x, \quad -\infty < x < \infty.$

Question 4. If $2x' = -\sqrt{3}x - y + 2$ and $2y' = x - \sqrt{3}y - 1$ are the equations for $\rho_{P,\theta}$, then find P and θ.

SOLUTION: Writing the equations in the form

$$\begin{aligned} x' &= -\frac{\sqrt{3}}{2}x - \frac{1}{2}y + 1 \\ y' &= \frac{1}{2}x - \frac{\sqrt{3}}{2}y - \frac{1}{2}, \end{aligned}$$

we see that these are the equations of a rotation $\rho_{P,\theta}$ about the point $P = (h,k)$ through the angle θ, where

$$\cos\theta = -\frac{\sqrt{3}}{2} \qquad \text{and} \qquad \sin\theta = \frac{1}{2},$$

so that $\theta = \dfrac{5\pi}{6}$.

To find the point $P = (h,k)$ we note that since P is a fixed point of the rotation $\rho_{P,\theta}$, then

$$\begin{aligned} h &= -\frac{\sqrt{3}}{2}h - \frac{1}{2}k + 1 \\ k &= \frac{1}{2}h - \frac{\sqrt{3}}{2}k - \frac{1}{2} \end{aligned}$$

with solution

$$h = 1 - \frac{\sqrt{3}}{4} \qquad \text{and} \qquad k = \frac{3}{4} - \frac{\sqrt{3}}{2}.$$

Question 5. If $x' = ax + by + c$ and $y' = bx - ay + d$ are equations for σ_m, then find the line m.

SOLUTION: We find the fixed points $P = (h, k)$ of σ_m by solving the system

$$h = ah + bk + c$$
$$k = bh - ak + d,$$

that is,

$$(1 - a)h - bk = c$$
$$-bh + (1 + a)k = d.$$

Since σ_m has infinitely many fixed points, this system must have infinitely many solutions, so the determinant of the coefficient matrix must be zero, that is, $a^2 + b^2 = 1$.

In this case, the fixed points $P = (h, k)$ must satisfy the equation $(a - 1)h + bk = c$, and the equation of the line m is $(a - 1)x + by + c = 0$, provided $a - 1 \neq 0$ or $b \neq 0$.

If $a = 1$ and $b = 0$, then first equation implies that $c = 0$, and the second equation gives $2k = d$, so that the equation of the line m in this case is $y = \dfrac{d}{2}$.

Question 6. Show that the equations for a glide reflection whose axis m passes through the origin with angle of inclination θ and whose translation is along m through r units, r measured positive from the origin into the first two quadrants or along the positive x-axis, and negative otherwise, are given by

$$x' = x \cos 2\theta + y \sin 2\theta + r \cos \theta$$
$$y' = x \sin 2\theta - y \cos 2\theta + r \sin \theta.$$

SOLUTION: The the glide reflection α is the product of a reflection in m and a translation along m by r units,

$$\alpha = \sigma_m \, \tau = \tau \, \sigma_m.$$

The equations of the translation τ are

$$x' = x + +r \cos \theta$$
$$y' = y + r \sin \theta$$

while the equations of the reflection are

$$x' = x \cos 2\theta + y \sin 2\theta$$
$$y' = x \sin 2\theta - y \cos 2\theta,$$

and therefore the equations of the glide reflection α are

$$x' = x \cos 2\theta + y \sin 2\theta + r \cos \theta$$
$$y' = x \sin 2\theta - y \cos 2\theta + r \sin \theta.$$

Question 7. If a and b are lines in the plane, show that the following are equivalent:

(a) $a = b$ or a and b are perpendicular,

(b) $\sigma_a \sigma_b = \sigma_b \sigma_a$,

(c) $\sigma_b(a) = a$,

(d) $(\sigma_b \sigma_a)^2 = \iota$,

(e) $\sigma_b \sigma_a$ is either the identity or a halfturn.

Solution:

(a) $\implies$ (b). If $a = b$, then $\sigma_a = \sigma_b$, so that $\sigma_a \sigma_b = \sigma_b \sigma_a$, while if a and b are perpendicular, then $\sigma_a \sigma_b$ is a rotation about the point of intersection P by an angle of π, that is, $\sigma_a \sigma_b = \sigma_P$. Similarly, $\sigma_b \sigma_a$ is a rotation about the point of intersection P by an angle of $-\pi$, that is, $\sigma_b \sigma_a = \sigma_P$, and $\sigma_a \sigma_b = \sigma_b \sigma_a$.

(b) $\implies$ (c). If $\sigma_a \sigma_b = \sigma_b \sigma_a$, then

$$\sigma_b \sigma_a \sigma_b = \sigma_a,$$

and since

$$\sigma_b \sigma_a \sigma_b = \sigma_{\sigma_b(a)},$$

then $a = \sigma_b(a)$.

(c) $\implies$ (d). If $\sigma_b(a) = a$ then

$$\sigma_b \sigma_a \sigma_b = \sigma_{\sigma_b(a)} = \sigma_a,$$

so that $(\sigma_b \sigma_a)^2 = \sigma_b \sigma_a \sigma_b \sigma_a = \iota$.

(d) $\implies$ (e). Suppose that $(\sigma_b \sigma_a)^2 = \iota$, if a and b are parallel, then $\sigma_b \sigma_a = \tau_{\overrightarrow{u}}$ is a translation, where $\overrightarrow{u}$ is perpendicular to a and b and $(\sigma_b \sigma_a)^2 = \tau_{2\overrightarrow{u}} = \iota$ implies that $\overrightarrow{u} = \overrightarrow{0}$, so that $\sigma_b \sigma_a = \iota$.

If a and b are not parallel, then they intersect at a point P, and $\sigma_b \sigma_a$ is a rotation about P by an angle θ. Since $(\sigma_b \sigma_a)^2 = \iota$ is a rotation about P by an angle $2\theta = 360$, so that $\theta = 180$, that is, $\sigma_b \sigma_a$ is a halfturn about P.

(e) $\implies$ (a). Suppose that $\sigma_b \sigma_a$ is either the identity or a halfturn.

If $\sigma_b \sigma_a$ is the identity, then $\sigma_b \sigma_a = \iota$ implies $\sigma_b = \sigma_a$, so that $a = b$.

If $\sigma_b \sigma_a$ is a halfturn, then it is a rotation about the point P of intersection of a and b by an angle of 180, so that the angle between a and b is 90, and a and b are perpendicular.

Question 8. If the isometry σ_P is a halfturn, show that given any two perpendicular lines m and n which intersect at the point P, we have $\sigma_P = \sigma_m \sigma_n$.

Solution: Given a halfturn σ_P about the point P, if m and n are perpendicular lines that intersect at P, then $\sigma_m \sigma_n$ is a rotation about P by an angle of 180, that is, $\sigma_m \sigma_n = \sigma_P$.