MATH 243 Winter 2008
Geometry II: Transformation Geometry
Solutions to Problem Set 4
Completion Date: Monday March 17, 2008

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Question 1. If ℓ, m, and n are the perpendicular bisectors of the sides $[AB]$, $[BC]$, and $[CA]$, respectively, of $\triangle ABC$, then

$$\alpha = \sigma_n \sigma_m \sigma_\ell$$

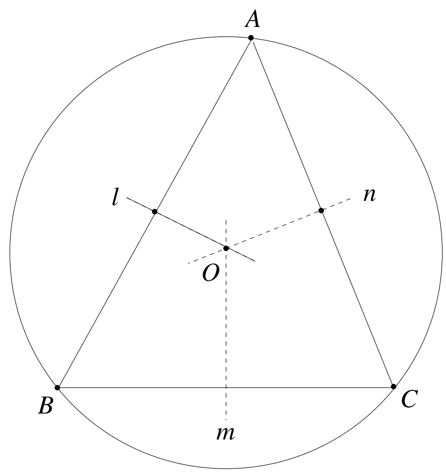
is a reflection in which line?

SOLUTION: Note that

$$\sigma_\ell(A) = B, \qquad \sigma_m(B) = C, \qquad \text{and} \qquad \sigma_n(C) = A,$$

so that

$$\sigma_n \sigma_m \sigma_\ell(A) = \sigma_n \sigma_m(B) = \sigma_n(C) = A.$$



Also, since the lines ℓ, m, and n are concurrent at the circumcenter O, then

$$\sigma_n \sigma_m \sigma_\ell(O) = \sigma_n \sigma_m(O) = \sigma_n(O) = O.$$

Finally, since

$$(\sigma_n \sigma_m \sigma_\ell)^2 = \sigma_n \sigma_m \sigma_\ell \sigma_n \sigma_m \sigma_\ell = \sigma_n \sigma_m \sigma_\ell \sigma_\ell \sigma_m \sigma_n = \sigma_n \sigma_m \sigma_m \sigma_n = \sigma_n \sigma_n = \iota.$$

Therefore the isometry $\alpha = \sigma_n \sigma_m \sigma_\ell$ is an involution and has two fixed points A and O, and is a reflection about the line through A and O.

Question 2. If $\sigma_n\sigma_m\sigma_\ell$ is a reflection, show that the lines ℓ, m, n are concurrent or have a common perpendicular.

Solution: Suppose that $\sigma_n\sigma_m\sigma_\ell$ is a reflection in the line p, say

$$\sigma_n\sigma_m\sigma_\ell = \sigma_p,$$

then

$$\sigma_m\sigma_\ell = \sigma_n\sigma_p,$$

is either a translation or a rotation.

If it is a rotation, say

$$\rho_{A,\theta} = \sigma_m\sigma_\ell = \sigma_n\sigma_p,$$

then

$$m \cap n = \{A\} = n \cap p,$$

so that the point A is on each of the lines ℓ, m, n, and p, and the lines ℓ, m, n are concurrent at A.

If it is a translation, say

$$\tau_{\overrightarrow{u}} = \sigma_m\sigma_\ell = \sigma_n\sigma_p,$$

then the lines m and ℓ are parallel, and the vector $\overrightarrow{u}$ is perpendicular to both ℓ and m. Also, the lines n and p are parallel, and the vector $\overrightarrow{u}$ is perpendicular to both n and p. Therefore the lines ℓ, m, and n have a common perpendicular $\overrightarrow{u}$.

Question 3. Find equations for lines m and n such that

$$\sigma_m\sigma_n(x, y) = (x + 2, y - 4).$$

Solution: Note that

$$\sigma_m\sigma_n(x, y) = (x + 2, y - 4) = \tau_{\overrightarrow{u}}(x, y)$$

is a translation by the vector $\overrightarrow{u} = \langle 2, -4\rangle$, so that m and n are parallel and perpendicular to $\overrightarrow{u}$.

We can take the direction vector of the lines m and n to be

$$\overrightarrow{v} = \langle 2, 1\rangle,$$

and let m be the line with this direction vector passing through the point $(0, 0)$, and n be the line passing through the point $(2, -4)$, that is,

$$\begin{aligned} x &= 2t \\ y &= t \end{aligned} \qquad (m)$$

and

$$\begin{aligned} x &= 2 + 2t \\ y &= -4 + t. \end{aligned} \qquad (n)$$

Question 4. Show that

$$\sigma_P \sigma_\ell \sigma_P \sigma_\ell \sigma_P \sigma_\ell \sigma_P$$

is a reflection in a line parallel to ℓ.

Solution: First we note that for any isometry α we have

$$\alpha \sigma_\ell \alpha^{-1} = \sigma_{\alpha(\ell)}. \tag{$*$}$$

To see this, since $\sigma_\ell^2 = \iota$, we have

$$\left(\alpha \sigma_\ell \alpha^{-1}\right)\left(\alpha \sigma_\ell \alpha^{-1}\right) = \alpha \sigma_\ell \sigma_\ell \alpha^{-1} = \alpha \alpha^{-1} = \iota,$$

and the isometry $\alpha \sigma_\ell \alpha^{-1}$ is an involution.

Also, if $Q \in \ell$, then $\sigma_\ell(Q) = Q$, and

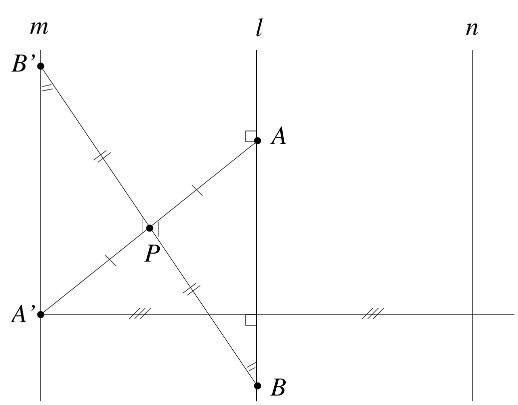
$$\alpha \sigma_\ell \alpha^{-1}(\alpha(Q)) = \alpha \sigma_\ell(Q) = \alpha(Q),$$

so that $\alpha(Q)$ is a fixed point of $\alpha \sigma_\ell \alpha^{-1}$ for each $Q \in \ell$.

Therefore $\alpha \sigma_\ell \alpha^{-1}$ is a reflection in $\alpha(\ell)$, that is, $\alpha \sigma_\ell \alpha^{-1} = \sigma_{\alpha(\ell)}$.

Now we show that if $\alpha = \sigma_P \sigma_\ell \sigma_P$, then $\alpha(\ell)$ is parallel to ℓ. We do this geometrically.

Let $m = \sigma_P(\ell)$, and note that the line m is parallel to ℓ from the SAS congruence theorem, as in the figure.



Finally, note that if $n = \sigma_\ell(m)$, then n is parallel to ℓ also, since m is parallel to ℓ. Therefore

$$\alpha(\ell) = \sigma_P \sigma_\ell \sigma_P(\ell) = \sigma_P \sigma_\ell(m) = \sigma_P(n)$$

is parallel to n and hence is also parallel to ℓ.

Therefore, if we let $\alpha = \sigma_P \sigma_\ell \sigma_P$, then

$$\sigma_P \sigma_\ell \sigma_P \sigma_\ell \sigma_P \sigma_\ell \sigma_P = \alpha \sigma_\ell \alpha^{-1}$$

is a reflection in the line $\alpha(\ell)$, which is parallel to ℓ.

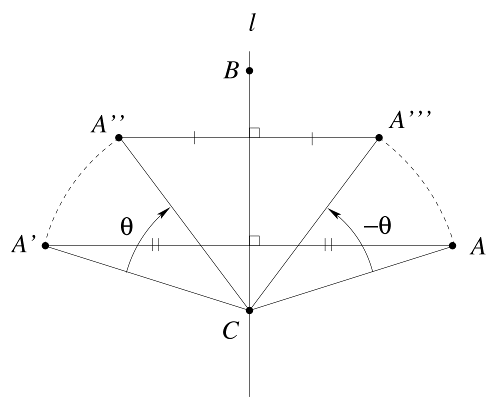
Question 5. Let C be a point on the line ℓ, show that

$$\sigma_\ell \rho_{C,\theta} \sigma_\ell = \rho_{C,-\theta}.$$

SOLUTION: Let A be an arbitrary point in $\mathcal{P}$, and let B be a second point on the line ℓ. Let

$$A' = \sigma_\ell(A), \qquad A'' = \rho_{C,\theta}(A'), \qquad \text{and} \qquad A''' = \sigma_\ell(A''),$$

as shown in the figure.



From the SAS congruence theorem, we have $CA = CA'$, and since $\rho_{C,\theta}$ is a rotation, we have $CA' = CA''$, and again by the SAS congruence theroem we have $CA'' = CA'''$, that is, the points A, A', A'', and A''' all lie on a circle with center at C and radius CA.

We have

$$\angle ACB - \angle A'''CB = |\,\theta\,| = \angle A'CB - \angle A''CB,$$

and the directed angles are

$$\angle ACA''' = -\theta = -\angle A'CA''.$$

Therefore

$$A''' = \sigma_\ell \rho_{C,\theta} \sigma_\ell(A) = \rho_{C,-\theta}(A),$$

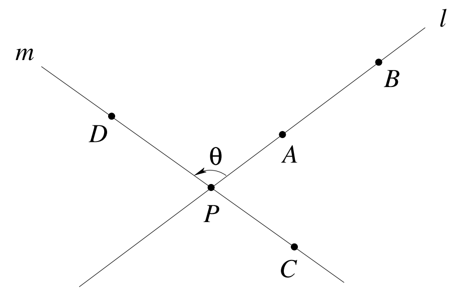
and since A is arbitrary, we have

$$\sigma_\ell \rho_{C,\theta} \sigma_\ell = \rho_{C,-\theta}.$$

Question 6. Given nonparallel lines AB and CD, show that there is a rotation ρ such that

$$\rho(AB) = CD.$$

SOLUTION: Let ℓ be the line through A and B and let m be the line through C and D, since ℓ and m are nonparallel, they intersect at a point P.



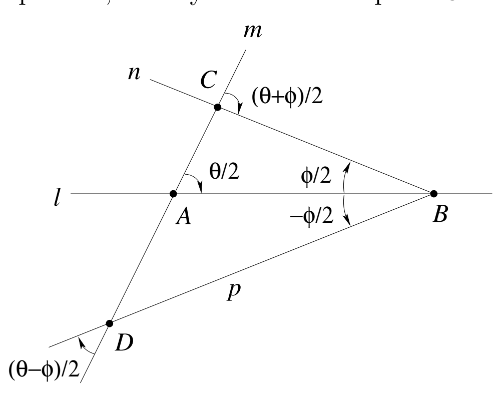
Let θ be the directed angle between ℓ and m, and let $\rho = \rho_{P,\theta}$ be the rotation about the point P through the directed angle θ, as in the figure, then $\rho(\ell) = m$, that is, $\rho(AB) = CD$.

Question 7. Show that if ρ_1, $\rho_2, \rho_2\rho_1$, and $\rho_2^{-1}\rho_1$ are rotations, then the center of ρ_1, $\rho_2\rho_1$, and $\rho_2^{-1}\rho_1$ are collinear.

SOLUTION: Let A and B be the centers of rotation for $\rho_1 = \rho_{A,\theta}$ and $\rho_2 = \rho_{B,\phi}$. We construct the center of rotation C for the product $\rho_2\rho_1$, that is, $\rho_{C,\theta+\phi} = \rho_{B,\phi}\,\rho_{A,\theta}$ as follows.

Let ℓ be the line through the points A and B, and let m be the line through A making a directed angle of $\theta/2$ with the line ℓ, as shown in the figure.

Next draw the line n through B making a directed angle $\phi/2$ with the line ℓ, as in the figure. Since $\rho_2\rho_1$ is a rotation, then m and n are not parallel, so they intersect at a point C as shown.



We construct the center of rotation D for the product $\rho_2^{-1}\rho_1$, that is, $\rho_{D,\theta-\phi} = \rho_{B,-\phi}\,\rho_{A,\theta}$ in the same way. Draw the line p through B making a directed angle $-\phi/2$ with the line ℓ, since $\rho_2^{-1}\rho_1$ is a rotation, then m and p are not parallel, so they intersect at a point D as shown.

The centers of rotation of ρ_1, $\rho_2\rho_1$, and $\rho_2^{-1}\rho_1$, that is, the points A, C, and D, are collinear and all lie on the line m.

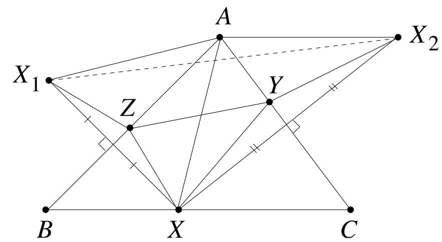
Question 8. In a given acute triangle, inscribe a triangle $\triangle PQR$ having minimum perimeter.

SOLUTION: This is **Fagnano's Problem**, and the following solution is due to L. Fejer.

Let $\triangle XYZ$ be inscribed in $\triangle ABC$ as in the figure below.

Reflect $\triangle AXZ$ in the line AB forming $\triangle AX_1Z$ (congruent to $\triangle AXZ$).

Similarly, Reflect $\triangle AXY$ in the line AC forming $\triangle AX_2Y$ (congruent to $\triangle AXY$).



From the figure, we have

$$XY + YZ + ZX = X_2Y + YZ + ZX_1 \geq X_1X_2 \qquad (*)$$

and since $\triangle AX_1X_2$ is isosceles, then

$$\tfrac{1}{2}X_1X_2 = AX_1 \sin \tfrac{1}{2}\angle X_1AX_2.$$

Also, since $AX_1 = AX$ and $\angle X_1AX_2 = 2\angle A$, then

$$X_1X_2 = 2AX \sin A,$$

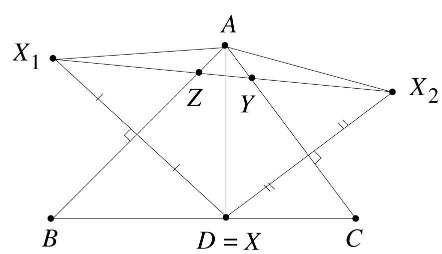
and from $(*)$, we have

$$XY + YZ + ZX \geq X_1X_2 = 2AX \sin A. \qquad (**)$$

However, if AD is the altitude from A, we obtain the general inequality

$$XY + YZ + ZX \geq X_1X_2 = 2AX \sin A \geq 2AD \sin A, \qquad (***)$$

which holds *for all inscribed triangles of* $\triangle ABC$.

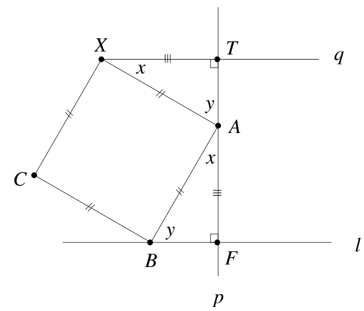


Equality holds in $(***)$ and the polygonal line joining X_1 and X_2 has minimum length if and only if the points X_1, Z, Y, and X_2 are collinear and AX is the altitude to side BC. Similarly, for the vertices B and C, so the incribed triangle $\triangle XYZ$ with minimum perimeter is the **orthic triangle**.

Question 9. Given a point A and two lines ℓ and m, construct a square $ABCD$ such that B lies on ℓ and D lies on m.

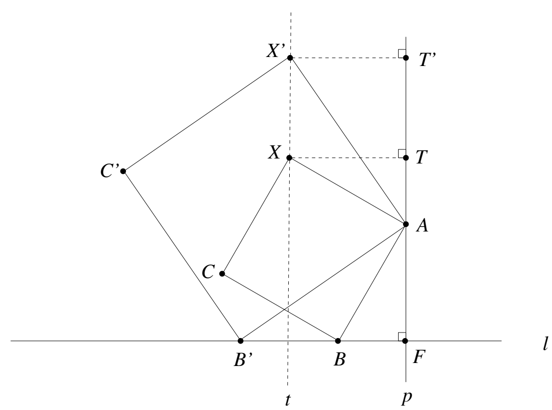
SOLUTION: (*Courtesy of Professor J. E. Lewis*) Given a line ℓ and a point A not on ℓ, we construct a square $ABCX$ with B on ℓ.

Draw the line p through A perpendicular to ℓ, hitting ℓ at F, and draw the line q through X perpendicular to p, hitting p at T, as in the figure.

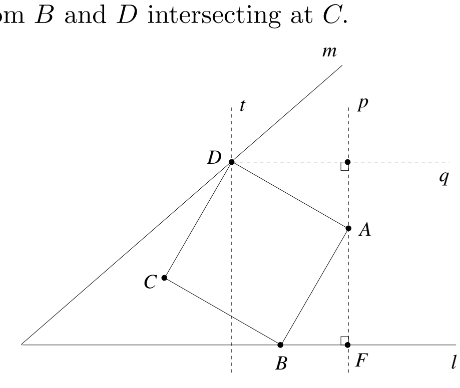


Since $AX = AB$, and $\angle AXT = \angle BAF$, then the right triangles $\triangle AFB$ and $\triangle XTA$ are congruent, and therefore $AF = XT$.

If A and ℓ are fixed, then as the vertex B of the square moves along the line ℓ, the distance of the vertex X from the line p is unchanged since $XT = AF$ is fixed, so that X moves along the line t which is parallel to and at a distance AF from p.



Now the construction is clear, given lines ℓ and m and a point A not on ℓ or m, drop a perpendicular p from A to ℓ, hitting ℓ at F. Next draw a line t parallel to p and a distance AF from p, hitting m at D. Now that the side AD of the square $ABCD$ has been constructed, we erect a perpendicular from A hitting ℓ at B, and then erect perpendiculars from B and D intersecting at C.

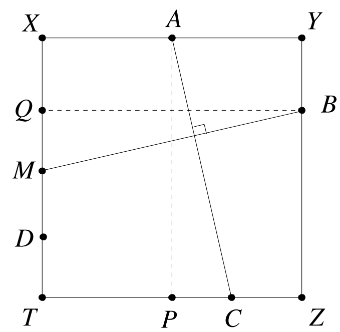


The case when A lies on either ℓ or m is left as an exercise.

Question 10. Given four distinct points, find a square such that each of the lines containing a side of the square passes through one of the four given points.

Hint: Given A, B, C, D we want to find the lines a, b, c, and d. Take P such that $\rho_{P,90}(B) = C$. Let $\rho_{P,90}(D) = E$, then take a to be AE.

SOLUTION: If the points A, B, C, D lie on the sides XY, YZ, ZT, TX of the square $XYZT$, as in the figure, and if the line through B perpendicular to AC meets the line XT at the point M, then since $\triangle ACP$ and $\triangle BMQ$ are congruent, then $BM = AC$.



Therefore, if we know four points A, B, C, D on the four sides of a square, then we can find one more point M on the side TX (or its extension) and draw the line TX passing through the two points D and M (assuming $D \neq M$).

Similarly, we can construct the line XY passing through A, the line YZ passing through B, and the line TZ passing through C.