MATH 243 Winter 2008
Geometry II: Transformation Geometry
Solutions to Problem Set 3
Completion Date: Friday February 29, 2008

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Question 1. Given a point O and a vector $\overrightarrow{u}$.

(a) Find the point Q such that
$$\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1} = \sigma_Q.$$

(b) What is the product $\sigma_O\tau_{\overrightarrow{u}}$?

SOLUTION:

(a) Note that
$$\left(\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1}\right)\left(\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1}\right) = \tau_{\overrightarrow{u}}\sigma_O{}^2\tau_{\overrightarrow{u}}^{-1} = \tau_{\overrightarrow{u}}\tau_{\overrightarrow{u}}^{-1} = \iota,$$
and since $\sigma_O \neq \iota$, then $\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1} \neq \iota$, so that $\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1}$ is an involutive isometry.
Also, we have
$$\left(\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1}\right)\tau_{\overrightarrow{u}}(O) = \tau_{\overrightarrow{u}}\sigma_O(O) = \tau_{\overrightarrow{u}}(O),$$
so that $\tau_{\overrightarrow{u}}(O)$ is a fixed point of $\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1}$.
Finally, note that if P is any other fixed point of $\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1}$, then
$$\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1}(P) = P,$$
so that
$$\sigma_O\tau_{\overrightarrow{u}}^{-1}(P) = \tau_{\overrightarrow{u}}^{-1}(P),$$
and $\tau_{\overrightarrow{u}}^{-1}(P)$ is a fixed point of σ_O, so that
$$\tau_{\overrightarrow{u}}^{-1}(P) = O,$$
that is, $P = \tau_{\overrightarrow{u}}(O)$ is the unique fixed point of $\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1}$.
Therefore, $\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1}$ is an involutive isometry with unique fixed point $\tau_{\overrightarrow{u}}(O)$, so that
$$\tau_{\overrightarrow{u}}\sigma_O\tau_{\overrightarrow{u}}^{-1} = \sigma_Q,$$
where $Q = \tau_{\overrightarrow{u}}(O)$.

(b) From the above, we see that
$$\sigma_O\tau_{\overrightarrow{u}} = \tau_{\overrightarrow{u}}\sigma_P$$
where $P = \tau_{\overrightarrow{u}}^{-1}(O)$.

Question 2. In the triangle $\triangle ABC$, show that G is the centroid if and only if

$$\sigma_G\sigma_C\sigma_G\sigma_B\sigma_G\sigma_A = \iota.$$

Solution: Let O be any point in $\mathcal{P}$, and note that since

$$\sigma_G\sigma_A = \tau_{2\overrightarrow{AG}}, \quad \sigma_G\sigma_B = \tau_{2\overrightarrow{BG}}, \quad \sigma_G\sigma_C = \tau_{2\overrightarrow{CG}},$$

then

$$\sigma_G\sigma_C\sigma_G\sigma_B\sigma_G\sigma_A = \iota$$

if and only if

$$\tau_{2(\overrightarrow{AG}+\overrightarrow{BG}+\overrightarrow{CG})} = \iota,$$

that is, if and only if

$$2\left(\overrightarrow{AG}+\overrightarrow{BG}+\overrightarrow{CG}\right) = \overrightarrow{0},$$

that is, if and only if

$$\overrightarrow{AO}+\overrightarrow{OG}+\overrightarrow{BO}+\overrightarrow{OG}+\overrightarrow{CO}+\overrightarrow{OG} = \overrightarrow{0},$$

that is, if and only if

$$\overrightarrow{OG} = \frac{1}{3}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}\right),$$

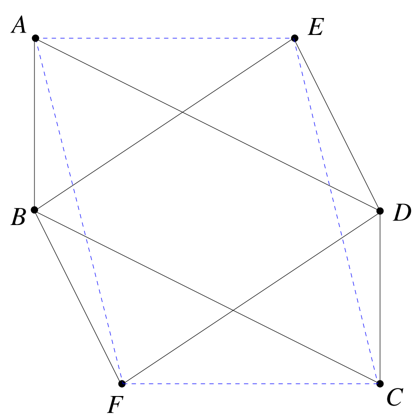
that is, if and only if G is the centroid of $\triangle ABC$.

Question 3. Prove using halfturns, that if $ABCD$ and $EBFD$ are parallelograms, then $EAFC$ is also a parallelogram.

Solution: Note that

$$\sigma_A\sigma_B\sigma_C\sigma_D = \iota \qquad \text{and} \qquad \sigma_D\sigma_F\sigma_B\sigma_E = \iota$$

since $ABCD$ and $EBFD$ are parallelograms.



Therefore,

$$\sigma_A\sigma_B\sigma_C\sigma_D\sigma_D\sigma_F\sigma_B\sigma_E = \iota^2 = \iota,$$

and since $\sigma_D^2 = \iota$, then

$$\sigma_A\sigma_B\sigma_C\sigma_F\sigma_B\sigma_E = \iota.$$

Since $\sigma_B\sigma_C\sigma_F = \sigma_F\sigma_C\sigma_B$, then

$$\sigma_A\sigma_F\sigma_C\sigma_B\sigma_B\sigma_E = \iota,$$

and since $\sigma_B^2 = \iota$, then

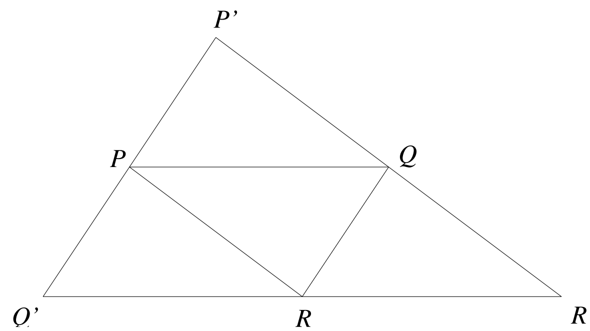
$$\sigma_A\sigma_F\sigma_C\sigma_E = \iota,$$

so that $EAFC$ is a parallelogram.

Question 4. Find all triangles such that three given noncollinear points are the midpoints of the sides of the triangle.

Hint: Given P, Q, R then $\sigma_R\sigma_Q\sigma_P$ fixes a vertex of a unique triangle $\triangle P'Q'R'$, as in the figure below.

SOLUTION: Suppose that $\triangle P'Q'R'$ is a triangle such that P, Q, and R are the respective midpoints of the sides $P'Q'$, $P'R'$, and $Q'R'$.



We have

$$\sigma_Q\sigma_R\sigma_P(P') = \sigma_Q\sigma_R(Q') = \sigma_Q(R') = P',$$

so that P' is the unique fixed point of the isometry (half turn) $\sigma_Q\sigma_R\sigma_P$.

Similarly,

$$\sigma_R\sigma_Q\sigma_P(Q') = \sigma_R\sigma_Q(P') = \sigma_R(R') = Q',$$

so that Q' is the unique fixed point of the isometry (half turn) $\sigma_R\sigma_Q\sigma_P$.

Finally,

$$\sigma_R\sigma_P\sigma_Q(R') = \sigma_R\sigma_P(P') = \sigma_R(Q') = R',$$

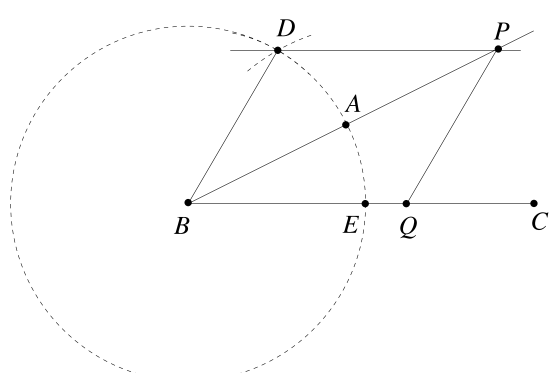
so that R' is the unique fixed point of the isometry (half turn) $\sigma_R\sigma_P\sigma_Q$.

Therefore the triangle $\triangle P'Q'R'$ is uniquely determined by the three noncollinear points P, Q, and R.

Question 5. Given $\angle ABC$, construct a point P on AB and a point Q on BC such that $PQ = AB$ and the line PQ intersects the line BC at an angle of $60°$.

Hint: Take a point D such that $[BD] \equiv [AB]$ and BD intersects BC at an angle of $60°$.

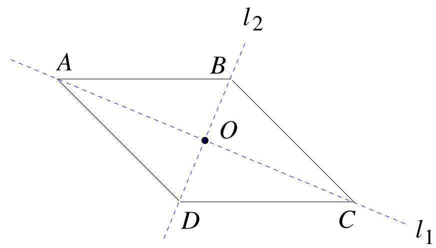
SOLUTION: Draw the circle $\mathcal{C}_1$ with center B and radius $|AB|$ hitting BC at E, then draw the circle $\mathcal{C}_2$ with center E and radius $|AB|$ intersecting $\mathcal{C}_1$ at D. Draw the line through D parallel to BC, hitting the line AB at P, and then mark off the length $|DP|$ on BC, hitting BC at Q.



The points P on AB and Q on BC are the desired points, since $DPQB$ is a parallelogram.

Question 6. What is the symmetry group of a rhombus that is not a square?

Solution: Let ℓ_1 and ℓ_2 be the diagonals of the nonsquare rhombus $ABCD$, as in the figure below.



Since the diagonals of a parallelogram bisect each other, and a parallelogram is a rhombus if and only if its diagonals are perpendicular, then the symmetries of the rhombus $ABCD$ are

$$\Sigma = \{ \iota, \sigma_O, \sigma_{\ell_1}, \sigma_{\ell_2} \}.$$

The Cayley table or multiplication table for the group of symmetries of the rhombus is given below.

$\cdot$	ι	σ_O	σ_{ℓ_1}	σ_{ℓ_2}
ι	ι	σ_O	σ_{ℓ_1}	σ_{ℓ_2}
σ_O	σ_O	ι	σ_{ℓ_2}	σ_{ℓ_1}
σ_{ℓ_1}	σ_{ℓ_1}	σ_{ℓ_2}	ι	σ_O
σ_{ℓ_2}	σ_{ℓ_2}	σ_{ℓ_1}	σ_O	ι

Question 7. Prove that if $\sigma_n \sigma_m$ fixes the point P and $m \neq n$, then the point P is on both lines m and n.

Solution: Suppose that P is a fixed point for the isometry $\sigma_n \sigma_m$, but P is not on both m and n, for example, suppose $P \notin m$.

Since $\sigma_n \sigma_m(P) = P$, then

$$\sigma_n^2 \sigma_m(P) = \sigma_n(P),$$

that is,

$$\sigma_m(P) = \sigma_n(P).$$

Now let

$$Q = \sigma_m(P) = \sigma_n(P),$$

and note that if $Q = P$, then $\sigma_m(P) = P$ and this implies that $P \in m$, which is a contradiction, therefore $Q \neq P$.

Thus m is the perpendicular bisector of the segment joining P and $Q = \sigma_m(P)$, and n is the perpendicular bisector of the segment joining P and $Q = \sigma_n(P)$, which contradicts the fact that $m \neq n$.

Therefore we must have $P \in m$. A similar argument shows that we must have $P \in n$ also.

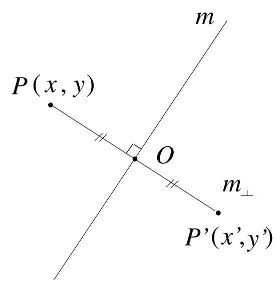
Question 8. Let m be a line with equation $2x + y = 1$. Find the equations of σ_m.

SOLUTION: If the equation of the line m is

$$ax + by + c = 0,$$

then the slope of m is $-\dfrac{a}{b}$, while the slope of a line $m_\perp$ which is perpendicular to m is $\dfrac{b}{a}$.

For $P \in \mathcal{P}$, let $P' = \sigma_m(P)$, and suppose that P has Cartesian coordinates (x, y), while P' has Cartesian coordinates (x', y').



Since P and P' are on $m_\perp$, we have

$$y' - y = \frac{b}{a}(x' - x),$$

and since the midpoint O of PP' has coordinates $\left(\dfrac{x + x'}{2}, \dfrac{y + y'}{2}\right)$, and O is on the line m, then

$$a\left(\frac{x + x'}{2}\right) + b\left(\frac{y + y'}{2}\right) + c = 0.$$

Solving these equations for x' and y', the equations of the reflection σ_m are given by

$$x' = x - \frac{2a}{a^2 + b^2}\left(ax + by + c\right)$$

$$y' = y - \frac{2b}{a^2 + b^2}\left(ax + by + c\right).$$

For the line $2x + y - 1 = 0$, we have $a = 2$, $b = 1$, and $c = -1$, so the equations of the reflection in this line are

$$x' = x - \frac{4}{5}(2x + y - 1)$$

$$y' = y - \frac{2}{5}(2x + y - 1).$$

Question 9. Suppose that the lines ℓ and m have equations $x + y = 0$ and $x - y = 1$, respectively. Find the equations for $\sigma_\ell \sigma_m$.

SOLUTION: Let $P = (x, y)$ and $P' = (x', y') = \sigma_m(P)$ and $P'' = (x'', y'') = \sigma_\ell(P')$, where for the lines m and ℓ we have

$$\begin{array}{lrlllll} m: & x - y - 1 = 0, & \text{so that} & a = 1, & b = -1, & c = -1 \\ \ell: & x + y = 0, & \text{so that} & a = 1, & b = 1, & c = 0. \end{array}$$

From the previous problem the equations of σ_m are

$$x' = y + 1$$
$$y' = x - 1,$$

while the equations of σ_ℓ are

$$x'' = -y'$$
$$y'' = -x'.$$

Therefore the equations of the isometry $\sigma_\ell \sigma_m$ are

$$x'' = -x + 1$$
$$y'' = -y - 1.$$

Question 10. Given triangles $\triangle ABC$ and $\triangle DEF$, where $\triangle ABC \equiv \triangle DEF$ where

$$A(0, 0),\ B(5, 0),\ C(0, 10),\ D(4, 2),\ E(1, -2),\ F(12, -4),$$

find the equations of the lines such that the product of reflections in these lines maps $\triangle ABC$ to $\triangle DEF$.

SOLUTION: Note first that

$$AB = DE = 5, \qquad AC = DF = 10, \qquad \text{and} \qquad BC = EF = \sqrt{125},$$

so that $\triangle ABC \equiv \triangle DEF$ by the SSS congruency theorem.

Let ℓ be the perpendicular bisector of the segment AD, since the midpoint of AD is the point

$$\frac{1}{2}(0 + 4, 0 + 2) = (2, 1),$$

and AD has slope $-\frac{1}{2}$, then the equation of ℓ is $y = -2x + 5$.

Therefore the reflection σ_ℓ has equations (see problem 8)

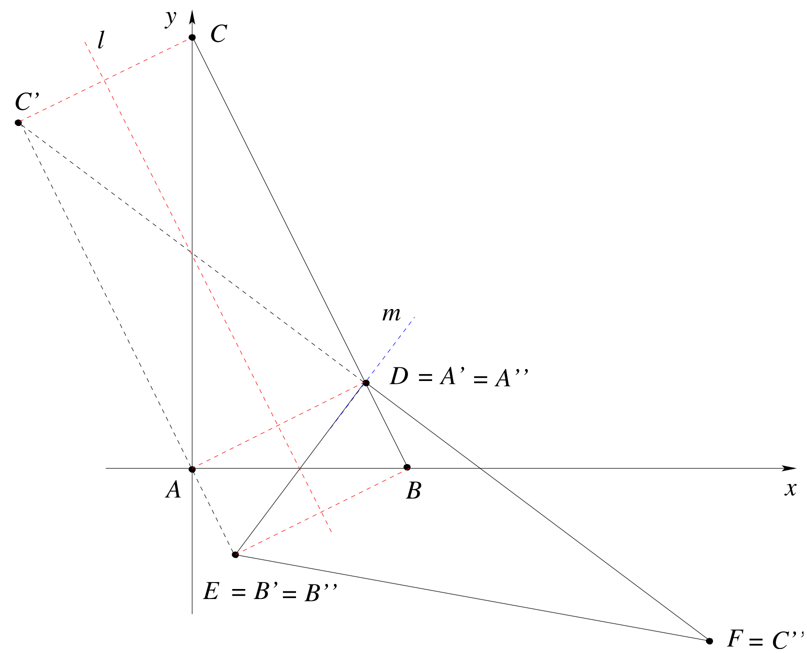
$$x' = -\frac{3}{5}x - \frac{4}{5}y + 4$$

$$y' = -\frac{4}{5}x + \frac{3}{5}y + 2.$$

The images of the vertices of $\triangle ABC$ under the reflection σ_ℓ are

$$A' = \sigma_\ell(A) = (4,2) = D, \qquad B' = \sigma_\ell(B) = (1,-2) = E, \qquad \text{and} \qquad C' = \sigma_\ell(C) = (-4,8)$$

as shown in the figure, and A' and B' are in the correct positions.



Now let m be the perpendicular pisector of the segment CC'', since the slope of m is the slope of DE, which is $\frac{4}{3}$, then the equation of m is $4x - 3y - 10 = 0$.

Therefore the reflection σ_m has equations (again, see problem 8)

$$x'' = -\frac{7}{25}x' + \frac{24}{25}y' + \frac{16}{5}$$

$$y'' = \frac{24}{25}x' - \frac{7}{25}y' - \frac{12}{5},$$

and the images of the vertices of $\triangle A'B'C'$ under the reflection σ_m are

$$A'' = \sigma_m(A') = (4,2) = D, \qquad B'' = \sigma_m(B') = (1,-2) = E, \qquad \text{and} \qquad C'' = \sigma_m(C') = (12,-4) = F.$$

Therefore the image of $\triangle ABC$ under the isometry

$$\alpha = \sigma_m \sigma_\ell$$

is the triangle $\triangle DEF$.