MATH 243 Winter 2008
Geometry II: Transformation Geometry
Solutions to Problem Set 2
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Question 1. Which of the mappings defined on the Cartesian plane $\mathcal{P}$ by the equations below are transformations?

(a) $\alpha(x, y) = (x^3, y^3)$ (b) $\beta(x, y) = (x, y^2)$

(c) $\gamma(x, y) = (3y, x + 2)$ (d) $\delta(x, y) = (x + y + 3, 2x + 2y - 1)$

SOLUTION:

(a) For any (x', y'), the equation $(x', y') = \alpha(x, y) = (x^3, y^3)$ has a unique solution

$$x = \sqrt[3]{x'}, \quad y = \sqrt[3]{y'},$$

and so α^{-1} exists and

$$\alpha^{-1}(x, y) = \left(\sqrt[3]{x}, \sqrt[3]{y}\right).$$

(b) If $(x_1, y_1) \in \mathbb{R}^2$, and $(x_2, y_2) \in \mathbb{R}^2$ with $x_1 = x_2$, but $y_2 = -y_1 \neq y_1$, then

$$\beta(x_1, y_1) = (x_1, y_1^2) = (x_2, y_2^2) = \beta(x_2, y_2),$$

so that β is not one-to-one, and so is not a transformation.

(c) Given any $(x', y') \in \mathbb{R}^2$, the system of equations

$$\begin{aligned} x' &= 3y \\ y' &= x + 2 \end{aligned}$$

has the unique solution

$$\begin{aligned} x &= y' - 2 \\ y &= \frac{1}{3}x' \end{aligned}$$

and therefore γ is a transformation.

(d) For any $(x', y') \in \mathbb{R}^2$, the system of equations

$$\begin{aligned} x' - 3 &= x + y \\ y' + 1 &= 2x + 2y \end{aligned}$$

has a solution if and only if $y' + 1 = 2x' - 6$, and therefore γ is not a transformation.

Question 2. Let $\alpha(x, y) = (x+1, y+2x)$ and $\beta = (x+y-1, y)$ be two mappings defined on the Cartesian plane $\mathcal{P}$.

(a) Show that α and β are transformations of $\mathcal{P}$.

(b) Find $\alpha\,\beta$ and $\beta\,\alpha$.

(c) Find α^{-1} and β^{-1}.

Solution:

(a) For $(x', y') \in \mathbb{R}^2$, the system of equations

$$\begin{aligned} x' &= x + 1 \\ y' &= y + 2x \end{aligned}$$

has a unique solution

$$\begin{aligned} x &= x' - 1 \\ y &= y' - 2x' + 2 \end{aligned} \tag{α^{-1}}$$

so that α is a transformation.
Also, for $(x', y') \in \mathbb{R}^2$, the system of equations

$$\begin{aligned} x' &= x + y - 1 \\ y' &= y \end{aligned}$$

has a unique solution

$$\begin{aligned} x &= x' - y' + 1 \\ y &= y' \end{aligned} \tag{β^{-1}}$$

so that β is a transformation.

(b) For any $(x, y) \in \mathbb{R}^2$, we have

$$\beta(\alpha(x, y)) = \beta(x+1, y+2x) = (3x+y, 2x+y),$$

and

$$\alpha(\beta(x, y)) = \alpha(x+y-1, y) = (x+y, 2x+3y-2).$$

(c) From (α) we have

$$\alpha^{-1}(x, y) = (x-1, y-2x+2),$$

while from (β) we have

$$\beta^{-1}(x, y) = (x-y+1, y).$$

Question 3.

(a) Find the image of the line $2x + 3y = 1$ under the affine transformation

$$\alpha(x, y) = (x + y + 1, x - y + 2).$$

(b) Find the fixed points of α.

Solution:

(a) If $(x', y') = \alpha(x, y) = (x + y + 1, x - y + 2)$, then

$$x = \frac{1}{2}x' + \frac{1}{2}y' - \frac{3}{2}$$
$$y = \frac{1}{2}x' - \frac{1}{2}y' + \frac{1}{2},$$

and the point (x, y) is on the line $2x + 3y = 1$ if and only if

$$x' + y' - 3 + \frac{3}{2}x' - \frac{3}{2}y' + \frac{3}{2} = 1,$$

that is,

$$\frac{5}{2}x' - \frac{1}{2}y' = \frac{5}{2}.$$

Therefore (x', y') is on the image of the line $2x + 3y = 1$ under the affine transformation $\alpha(x, y)$ if and only if

$$5x' - y' = 5.$$

(b) If (x_0, y_0) is a fixed point of α, then

$$(x_0, y_0) = \alpha(x_0, y_0) = (x_0 + y_0 + 1, x_0 - y_0 + 2),$$

so that $y_0 = -1$ and $x_0 = 2y_0 - 2 = -4$, and $(-4, -1)$ is the unique fixed point of α.

Question 4.

(a) Prove that any affine transformation is a collineation.

(b) Show that $\alpha(x, y) = (2x^3 + 1, y^3)$ is a transformation of the plane but is not a collineation.

Solution:

(a) Let $(x', y') = \alpha(x, y) = (ax + by + c, dx + ey + f)$ be an affine transformation, then $\Delta = ae - bd \neq 0$, and

$$x = \frac{1}{\Delta}\left[e(x' - c) - b(y' - f)\right]$$
$$y = -\frac{1}{\Delta}\left[d(x' - c) - a(y' - f)\right],$$

so that (x, y) is on a line $Ax + By + C = 0$ if and only if

$$\frac{A}{\Delta}\left[e(x' - c) - b(y' - f)\right] - \frac{B}{\Delta}\left[d(x' - c) - a(y' - f)\right] + C = 0,$$

and the image of a line $Ax + By + C = 0$ is also a line $A'x' + B'y' + C' = 0$, where

$$A' = \frac{1}{\Delta}(Ae - Bd), \quad B' = -\frac{1}{\Delta}(Ab - Ba), \quad C' = \frac{1}{\Delta}(-Aec + Abf + Bcd - Baf) + C.$$

(b) For any $(x', y') \in \mathbb{R}^2$, the system of equations

$$\begin{aligned} x' &= 2x^3 + 1 \\ y' &= y^3 \end{aligned}$$

has a unique solution

$$\begin{aligned} x &= \left(\frac{1}{2}(x' - 1)\right)^{1/3} \\ y &= y'^{1/3}, \end{aligned}$$

and α is a transformation. Clearly the point (x, y) is on the line $ax + by + c = 0$ if and only if

$$a\left(\frac{1}{2}(x' - 1)\right)^{1/3} + by'^{1/3} + c = 0$$

and $\{(x', y') \,|\, ax + by + c = 0\}$ is **not** the equation of a line in the plane, so α is **not** a collineation.

Question 5. Let α and β be two involutive transformations of the Cartesian plane $\mathcal{P}$.

(a) Prove that $\alpha\,\beta$ is involutive if and only if $\alpha\,\beta = \beta\,\alpha$.

(b) Assume that α, β, ι are distinct transformations such that

$$\alpha\,\beta = \beta\,\alpha = \gamma.$$

Let $\Gamma = \{\iota,\ \alpha,\ \beta,\ \gamma\}$. Prove that Γ is a commutative subgroup of $\mathcal{G}$, the group of all transformations on the plane $\mathcal{P}$ (construct the multiplication table).

SOLUTION:

(a) If α and β are involutions, then

$$\iota = (\alpha\beta)^2 = \alpha\beta\alpha\beta$$

if and only if

$$\alpha^2\beta\alpha\beta^2 = \alpha\iota\beta = \alpha\beta,$$

that is, if and only if

$$\iota\beta\alpha\iota = \alpha\beta,$$

that is, if and only if

$$\beta\alpha = \alpha\beta.$$

(b) Note that $\gamma^2 = (\alpha\beta)^2 = \iota$ from part (a), so that γ is an involution. Therefore

$$\iota^{-1} = \iota, \qquad \alpha^{-1} = \alpha, \qquad \beta^{-1} = \beta, \qquad \text{and} \qquad \gamma^{-1} = \gamma,$$

so that Γ is closed under taking inverses.

Also,

$$\alpha\beta = \beta\alpha = \gamma, \quad \alpha\gamma = \alpha^2\beta = \beta = \beta\alpha^2 = \gamma\alpha, \quad \beta\gamma = \beta^2\alpha = \alpha = \alpha\beta^2 = \gamma\beta,$$

and Γ is closed under multiplication and multiplication is commutative. Therefore Γ is a subgroup of $\mathcal{G}$.

Question 6. Let $\alpha(x, y) = (ax + by, cx + dy)$ be an affine transformation of $\mathcal{P}$. Prove that α is an involution if and only if

$$\begin{aligned} a^2 + bc &= 1 \\ ab + bd &= 0 \\ ac + cd &= 0 \\ bc + d^2 &= 1. \end{aligned}$$

Note: The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called the matrix of the transformation α. The conditions above say that α is an involution if and only if $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

SOLUTION: For each (x', y'), the system of equations

$$\begin{aligned} x' &= ax + by \\ y' &= cx + dy \end{aligned}$$

has a unique solution if and only if the coefficient matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ has a nonzero determinant, that is if and only if $ad - bc \neq 0$.

If we write $(x', y') = \alpha(x, y) = (ax + by, cx + dy)$ in vector form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

and if $(x'', y'') = \alpha(x', y')$, then

$$\begin{pmatrix} x'' \\ y'' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 \begin{pmatrix} x \\ y \end{pmatrix}.$$

Therefore α is an involution if and only if

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

that is, if and only if

$$\begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

that is, if and only if

$$\begin{aligned} a^2 + bc &= 1 \\ ab + bd &= 0 \\ ac + cd &= 0 \\ bc + d^2 &= 1. \end{aligned}$$

Question 7. Let α be an isometry of $\mathcal{P}$ which admits an invariant line ℓ (that is, $\alpha(\ell) = \ell$) and a fixed point $P \in \mathcal{P}$. Prove that there is a point $Q \in \ell$ such that $\alpha(Q) = Q$ and a line ℓ' through P such that $\alpha(\ell') = \ell'$.

SOLUTION: If $P \in \ell$, then we may take $Q = P$ so that $\alpha(Q) = \alpha(P) = P = Q$, and ℓ' the line through Q which is perpendicular to ℓ. Since α is an isometry, it preserves perpendicularity so that $\alpha(\ell')$ is perpendicular to $\alpha(\ell) = \ell$, so that $\alpha(\ell')$ is parallel to ℓ'. Since they both pass through $P = \alpha(P)$, then $\alpha(\ell') = \ell'$.

If P is not on ℓ, drop a perpendicular from P to the line ℓ, hitting ℓ at Q. Now, $\alpha(\ell) = \ell$ so that $\alpha(Q) \in \ell$, and α is an isometry, so that

$$d(P, Q) = d(\alpha(P), \alpha(Q)) = d(P, \alpha(Q)).$$

If $Q \neq \alpha(Q)$, then since the hypotenuse of the right triangle $\triangle P\,Q\,\alpha(Q)$ is the longest side of this right triangle, we have $d(P, Q) < d(P, \alpha(Q))$, which is a contradiction, therefore, $\alpha(Q) = Q$.

Now let ℓ' be the line passing through the points $P = \alpha(P)$ and $Q = \alpha(Q)$, since α preserves perpendicularity, then $\alpha(\ell')$ is perpendicular to $\alpha(\ell) = \ell$, so that ℓ' is parallel to $\alpha(\ell')$ and both these lines pass through P and Q, so that $\alpha(\ell') = \ell'$.

Question 8. If a circle is invariant under the isometry α then its center is a fixed point of α.

SOLUTION: Let P be any point on the circle $\mathcal{C}$ with center O and radius a, if $\mathcal{C}$ is invariant under the isometry α, then $\alpha(P)$ is on $\mathcal{C}$ for every P on $\mathcal{C}$. Therefore,

$$d(O, P) = d(\alpha(O), \alpha(P)) = a$$

for every $P \in \mathcal{C}$.

Since α maps the circle $\mathcal{C}$ onto the circle, then given any point Q on the circle, there exists a point P on the circle such that $Q = \alpha(P)$, so that

$$d(\alpha(O), Q) = d(\alpha(O), \alpha(P)) = a,$$

that is,

$$d(\alpha(O), Q) = a$$

for each Q on the circle $\mathcal{C}$. This says that each point on the circle is equidistant from the point $\alpha(O)$, that is, $\alpha(O) = O$, the center of the circle. Thus $\alpha(O) = O$ and the center O is a fixed point of the isometry α.

Question 9. Let $\alpha \neq \iota$ be an involutive isometry, show that α has at least one fixed point.

SOLUTION: Suppose that α is an involutive isometry and let P be any point in the plane $\mathcal{P}$, if $\alpha(P) = P$ then we are done.

If P is not a fixed point of α, let M be the midpoint of the segment from P to $\alpha(P)$, then

$$d(M, P) = d(M, \alpha(P)),$$

and

$$d(\alpha(M), \alpha(P)) = d(\alpha(M), \alpha^2(P)) = d(\alpha(M), P),$$

therefore, $\alpha(M)$ is on the perpendicular bisector of the segment from P to $\alpha(P)$.

Since α maps the line $\ell = \ell_{P\alpha(P)}$ onto itself, then $\alpha(M)$ is also on the the line ℓ, so that $\alpha(M) = M$ and M is a fixed point of α.

Question 10. Let α be an isometry of $\mathcal{P}$ and let ℓ be the perpendicular bisector of the segment $[AB]$. Prove that $\alpha(\ell)$ is the perpendicular bisector of the segment $[\alpha(A)\,\alpha(B)]$.

SOLUTION: Let M be the midpoint of the segment $[A, B]$, then

$$d(M, A) = d(M, B)$$

and since α is an isometry, then

$$d(\alpha(M), \alpha(A)) = d(M, A) = d(M, B) = d(\alpha(M), \alpha(B)),$$

so that $\alpha(M)$ is on the perpendicular bisector of $[\alpha(A), \alpha(B)]$.

Similarly, if P is any other point on the perpendicular bisector of the segment $[AB]$, then

$$d(\alpha(A), \alpha(P)) = d(A, P) = d(B, P) = d(\alpha(B), \alpha(P)),$$

so that $\alpha(P)$ is also on the perpendicular bisector of $[\alpha(A), \alpha(B)]$. Since ℓ is the line passing through M and P, then $\alpha(\ell)$ is the line passing through $\alpha(M)$ and $\alpha(P)$, that is, $\alpha(\ell)$ is the perpendicular bisector of the segment $[\alpha(A)\,\alpha(B)]$.