MATH 243 Winter 2008
Geometry II: Transformation Geometry
Solutions to Problem Set 1
Completion Date: Monday January 21, 2008

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Question 1. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors, both parallel to a line ℓ.

(a) Show that $\mathbf{u}+\mathbf{v}$ is parallel to ℓ.

(b) Show that $k\mathbf{u}$ is parallel to ℓ for each $k \in \mathbb{R}$, $k \neq 0$.

SOLUTION:

(a) Let A be an arbitrary point in the Euclidean point space $\mathcal{E}$, and apply the vector $\overrightarrow{u}$ in the translation space $\mathcal{V}$ of $\mathcal{E}$ to the point A to get the point B such that $\overrightarrow{AB} = \overrightarrow{u}$. Next apply the vector $\overrightarrow{v}$ to the point B to get the point C such that $\overrightarrow{BC} = \overrightarrow{v}$, since $\overrightarrow{u}$ and $\overrightarrow{v}$ are parallel to ℓ, then the directed line segments (A,B) and (B,C) are also parallel to ℓ, and therefore have the same support line, thus, the points A, B, and C are collinear. From the definition of vectors in $\mathcal{V}$, this means that $\overrightarrow{u} + \overrightarrow{v} = \overrightarrow{AC}$ is parallel to ℓ also.

(b) Assume that $k > 0$, and let A an arbitrary point in the Euclidean point space $\mathcal{E}$, and apply the vector $\overrightarrow{u}$ in the translation space $\mathcal{V}$ of $\mathcal{E}$ to the point A to get the point B such that $\overrightarrow{AB} = \overrightarrow{u}$. Let C be the point on the support line of the directed line segment (A,B) such that $|AC| = k|AB|$. From the definition of scalar multiplication In $\mathcal{V}$, we have $\overrightarrow{AC} = k\overrightarrow{u}$, and the directed line segment (A,C) is parallel to ℓ, that is, $k\overrightarrow{u} = \overrightarrow{AC}$ is parallel to ℓ also.

Question 2. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors, both parallel to a plane Π.

(a) Show that $\mathbf{u}+\mathbf{v}$ is parallel to Π.

(b) Show that $k\mathbf{u}$ is parallel to Π for each $k \in \mathbb{R}$, $k \neq 0$.

SOLUTION:

(a) Let $(\overrightarrow{e}_1, \overrightarrow{e}_2)$ be a planar direction for the plane Π, then we can write

$$\overrightarrow{u} = a\overrightarrow{e}_1 + b\overrightarrow{e}_2 \qquad \text{and} \qquad \overrightarrow{v} = c\overrightarrow{e}_1 + d\overrightarrow{e}_2$$

for some scalars a, b, c, d, so that

$$\overrightarrow{u} + \overrightarrow{v} = (a+c)\overrightarrow{e}_1 + (b+d)\overrightarrow{e}_2,$$

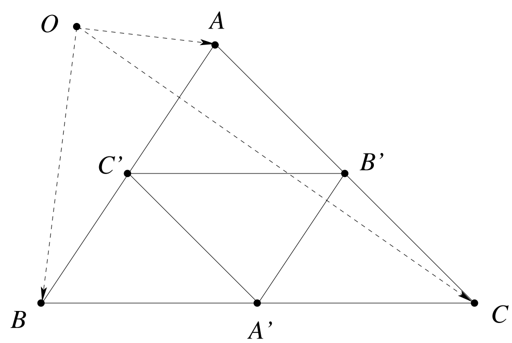
and $\overrightarrow{u} + \overrightarrow{v}$ is also parallel to the plane Π.

(b) Similarly, if $k \neq 0$, then $k\overrightarrow{u} = ka\overrightarrow{e}_1 + kb\overrightarrow{e}_2$, so that $k\overrightarrow{u}$ is parallel to the plane Π.

Question 3. Given an arbitrary point O, let A', B', C', be the midpoints of the sides BC, AC, and AB of $\triangle ABC$, show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} = \overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'}.$$

Solution: In the figure



we have

$$\overrightarrow{OA'} = \overrightarrow{OB} + \frac{1}{2}\overrightarrow{BC} = \overrightarrow{OB} + \frac{1}{2}(\overrightarrow{OC} - \overrightarrow{OB}) = \frac{1}{2}(\overrightarrow{OB} + \overrightarrow{OC}),$$

and similarly

$$\overrightarrow{OB'} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OC}) \qquad \text{and} \qquad \overrightarrow{OC'} = \frac{1}{2}(\overrightarrow{OA} + \overrightarrow{OB}),$$

and therefore

$$\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'} = \overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}.$$

Question 4. Given $\triangle A\,B\,C$ and $\triangle A'\,B'\,C'$, let G and G' be their centroids, respectively. Show that

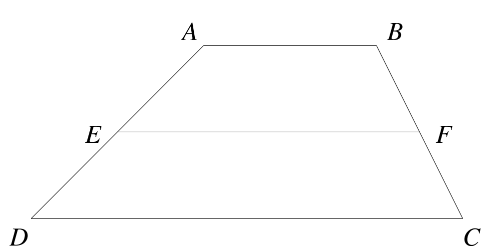
$$\overrightarrow{GG'} = \frac{1}{3}\left(\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}\right).$$

Solution: If we introduce the point $O \in \mathcal{E}$, then

$$\begin{aligned}\overrightarrow{GG'} = \overrightarrow{OG'} - \overrightarrow{OG} &= \frac{1}{3}\left(\overrightarrow{OA'} + \overrightarrow{OB'} + \overrightarrow{OC'}\right) - \frac{1}{3}\left(\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC}\right) \\ &= \frac{1}{3}\left(\overrightarrow{OA'} - \overrightarrow{OA}\right) + \frac{1}{3}\left(\overrightarrow{OB'} - \overrightarrow{OB}\right) + \frac{1}{3}\left(\overrightarrow{OC'} - \overrightarrow{OC}\right) = \frac{1}{3}\left(\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}\right).\end{aligned}$$

Question 5. Show using vectors that the segment joining the midpoints of the nonparallel sides of a trapezoid is parallel to either base and congruent to half the sum of the bases.

Solution: In the figure

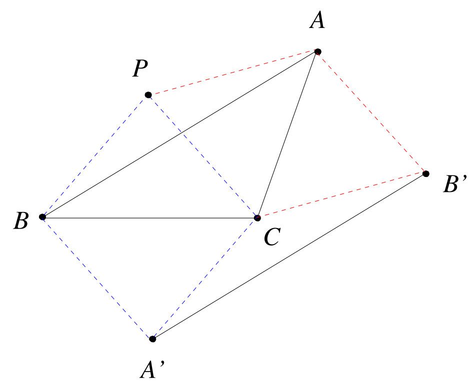


we have $\overrightarrow{EF} = \overrightarrow{EA} + \overrightarrow{AB} + \overrightarrow{BF}$ and $\overrightarrow{EF} = \overrightarrow{ED} + \overrightarrow{DC} + \overrightarrow{CF}$ so that $2\overrightarrow{EF} = \overrightarrow{AB} + \overrightarrow{DC}$, since $\overrightarrow{EA} + \overrightarrow{ED} = \overrightarrow{0}$ and $\overrightarrow{BF} + \overrightarrow{CF} = \overrightarrow{0}$, therefore

$$\overrightarrow{EF} = \frac{1}{2}\left(\overrightarrow{AB} + \overrightarrow{DC}\right).$$

Question 6. If the point P lies in the plane of $\triangle ABC$, but is distinct from the vertices of the triangle, and the parallelograms $PBA'C$, $PCB'A$, $PAC'B$ are completed, show that the segments $[AA']$, $[BB']$, and $[CC']$ bisect each other.

SOLUTION: In the figure we have completed the parallelograms $PBA'C$ and $PCB'A$.



Let Q and R be the midpoints of the segements AA' and BB', respectively (not shown). then

$$\begin{aligned}\overrightarrow{PQ} &= \overrightarrow{PA} + \frac{1}{2}\overrightarrow{AA'}\\ &= \overrightarrow{PA} + \frac{1}{2}\left(\overrightarrow{PA'} - \overrightarrow{PA}\right)\\ &= \frac{1}{2}\overrightarrow{PA} + \frac{1}{2}\left(\overrightarrow{PB} + \overrightarrow{BA'}\right)\\ &= \frac{1}{2}\overrightarrow{PA} + \frac{1}{2}\left(\overrightarrow{PB} + \overrightarrow{PC}\right),\end{aligned}$$

and

$$\overrightarrow{PQ} = \frac{1}{2}\left(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}\right).$$

Also,

$$\begin{aligned}\overrightarrow{PR} &= \overrightarrow{PB} + \frac{1}{2}\overrightarrow{BB'}\\ &= \overrightarrow{PB} + \frac{1}{2}\left(\overrightarrow{PB'} - \overrightarrow{PB}\right)\\ &= \frac{1}{2}\overrightarrow{PB} + \frac{1}{2}\left(\overrightarrow{PC} + \overrightarrow{CB'}\right)\\ &= \frac{1}{2}\overrightarrow{PB} + \frac{1}{2}\left(\overrightarrow{PC} + \overrightarrow{PA}\right),\end{aligned}$$

and

$$\overrightarrow{PR} = \frac{1}{2}\left(\overrightarrow{PA} + \overrightarrow{PB} + \overrightarrow{PC}\right).$$

Therefore $Q = R$, and the line segments AA' and BB' bisect each other. Similarly, the line segments AA' and CC' bisect each other.

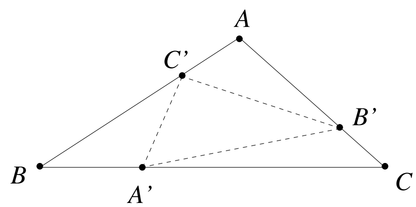
Question 7. Suppose that the points A', B', C' lie on the sides $[BC]$, $[CA]$, and $[AB]$ of $\triangle ABC$, respectively, and

$$\frac{A'B}{A'C} = \frac{B'C}{B'A} = \frac{C'A}{C'B}.$$

(a) Show that the centroids of $\triangle A'B'C'$ and $\triangle ABC$ coincide.

(b) If the parallelograms $AC'C''C$ and $AB'B''B$ are completed, show that $B'C''$ and $C'B''$ are parallel to the median of $\triangle ABC$ through A.

Solution: In the figure



let A', B', C' be points on the sides $[BC]$, $[CA]$, and $[AB]$ of $\triangle ABC$, respectively, such that

$$\frac{A'B}{A'C} = \frac{B'C}{B'A} = \frac{C'A}{C'B} = k,$$

where $k > 0$ (if $k = 0$, then the triangles coincide).

(a) We know from Question 4 that if G and G' are the centroids of $\triangle ABC$ and $\triangle A'B'C'$, respectively, then

$$\overrightarrow{GG'} = \frac{1}{3}\left(\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'}\right),$$

and in order to show that the centroids coincide, we only need to show that

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \overrightarrow{0}.$$

Note that since $\dfrac{A'B}{A'C} = k$, then $\overrightarrow{BA'} = k\overrightarrow{A'C}$, so that

$$\overrightarrow{BC} = \overrightarrow{BA'} + \overrightarrow{A'C} = \frac{k+1}{k} \cdot \overrightarrow{BA'}$$

and

$$\overrightarrow{AA'} = \overrightarrow{AB} + \overrightarrow{BA'} = \overrightarrow{AB} + \frac{k}{k+1}\overrightarrow{BC}. \tag{1}$$

Also, note that since $\dfrac{B'C}{B'A} = k$, then $\overrightarrow{CB'} = k\overrightarrow{B'A}$, so that

$$\overrightarrow{CA} = \overrightarrow{CB'} + \overrightarrow{B'A} = \frac{k+1}{k} \cdot \overrightarrow{CB'}$$

and

$$\overrightarrow{BB'} = \overrightarrow{BC} + \overrightarrow{CB'} = \overrightarrow{BC} + \frac{k}{k+1}\overrightarrow{CA}. \tag{2}$$

Finally, note that since $\dfrac{C'A}{C'B} = k$, then $\overrightarrow{AC'} = k\overrightarrow{C'B}$, so that

$$\overrightarrow{AB} = \overrightarrow{AC'} + \overrightarrow{C'B} = \frac{k+1}{k} \cdot \overrightarrow{AC'}$$

and

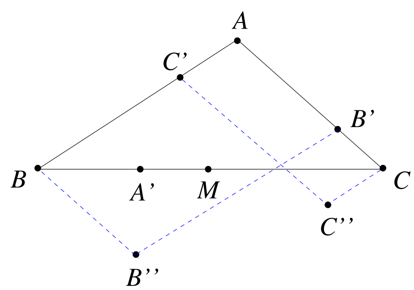
$$\overrightarrow{CC'} = \overrightarrow{CA} + \overrightarrow{AC'} = \overrightarrow{CA} + \frac{k}{k+1}\overrightarrow{AB}. \tag{3}$$

Adding (1), (2), and (3), we have

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \left(\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA}\right) + \frac{k}{k+1}\left(\overrightarrow{BC} + \overrightarrow{CA} + \overrightarrow{AB}\right) = \overrightarrow{0}$$

since $\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{0}$.

(b) In the figure below we have completed the parallelograms $AC'C''C$ and $AB'B''B$, and let M be the midpoint of the side BC. We want to show that $C'B''$ and $B'C''$ are both parallel to AM.



Let $t = \dfrac{AB'}{B'C}$, so that $\overrightarrow{AB'} = t\overrightarrow{B'C}$, then

$$\overrightarrow{AC} = \overrightarrow{AB'} + \overrightarrow{B'C} = (1+t)\overrightarrow{B'C},$$

and

$$\overrightarrow{B'C} = \frac{1}{1+t}\overrightarrow{AC},$$

that is,

$$\overrightarrow{AB'} = \overrightarrow{AC} - \overrightarrow{B'C} = \frac{t}{1+t}\overrightarrow{AC}. \tag{4}$$

Also, $t = \dfrac{AB'}{B'C} = \dfrac{C'B}{AC'}$, so that $\overrightarrow{C'B} = t\overrightarrow{AC'}$, and

$$\overrightarrow{AB} = \overrightarrow{AC'} + \overrightarrow{C'B} = (1+t)\overrightarrow{AC'}$$

so that

$$\overrightarrow{AC'} = \frac{1}{1+t}\overrightarrow{AB}.$$

Therefore,

$$\overrightarrow{AC''} = \overrightarrow{AC'} + \overrightarrow{C'C''} = \overrightarrow{AC'} + \overrightarrow{AC} = \frac{1}{1+t}\overrightarrow{AB} + \overrightarrow{AC}. \tag{5}$$

Subtracting (4) from (5) we have

$$\overrightarrow{B'C''} = \overrightarrow{AC''} - \overrightarrow{AB'} = \frac{1}{1+t}\left(\overrightarrow{AB} + \overrightarrow{AC}\right) = \frac{2}{1+t}\overrightarrow{AM},$$

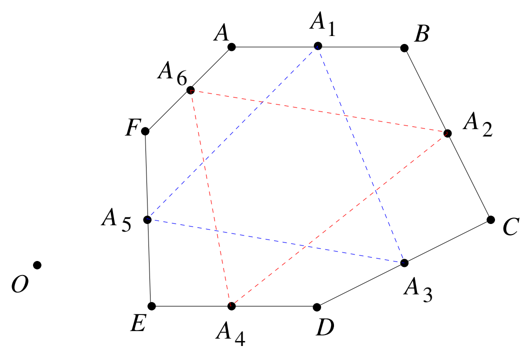
and $B'C''$ is parallel to the median through A. Similarly, $C'B''$ is parallel to the median through A.

Question 8. If A_1, A_2, A_3, A_4, A_5, A_6 are the midpoints of consecutive sides of a hexagon, show that $\triangle A_1A_3A_5$ and $\triangle A_2A_4A_6$ have the same centroid.

SOLUTION: Given the hexagon $ABCDEF$ in the figure, let

$$\begin{aligned} A_1 &= \text{ midpoint of AB}, & A_2 &= \text{ midpoint of BC}, & A_3 &= \text{ midpoint of CD}, \\ A_4 &= \text{ midpoint of DE}, & A_5 &= \text{ midpoint of EF}, & A_6 &= \text{ midpoint of FA}, \end{aligned}$$

and let O be an arbitrary point in the Euclidean point space $\mathcal{E}$.

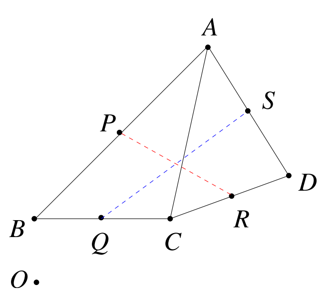


Let G_{odd} be the centroid of $\triangle A_1A_3A_5$ and G_{even} be the centroid of $\triangle A_2A_4A_6$, then

$$\begin{aligned} \overrightarrow{OG}_{\text{odd}} &= \frac{1}{3}\left(\overrightarrow{OA_1} + \overrightarrow{OA_3} + \overrightarrow{OA_5}\right) \\ &= \frac{1}{3}\left(\frac{\overrightarrow{OA} + \overrightarrow{OB}}{2}\right) + \frac{1}{3}\left(\frac{\overrightarrow{OC} + \overrightarrow{OD}}{2}\right) + \frac{1}{3}\left(\frac{\overrightarrow{OE} + \overrightarrow{OF}}{2}\right) \\ &= \frac{1}{3}\left(\frac{\overrightarrow{OA} + \overrightarrow{OF}}{2}\right) + \frac{1}{3}\left(\frac{\overrightarrow{OB} + \overrightarrow{OC}}{2}\right) + \frac{1}{3}\left(\frac{\overrightarrow{OD} + \overrightarrow{OE}}{2}\right) \\ &= \frac{1}{3}\left(\overrightarrow{OA_6} + \overrightarrow{OA_2} + \overrightarrow{OA_4}\right) \\ &= \overrightarrow{OG}_{\text{even}}. \end{aligned}$$

Question 9. If P, Q, R, S are the midpoints of the edges AB, BC, CD, DA, respectively, of a tetrahedron $ABCD$, show that PR and QS bisect each other at the centroid of the tetrahedron.

SOLUTION: In the figure, let O be an arbitrary point in the Euclidean point space $\mathcal{E}$, and let G be the centroid of the tetrahedron.



From the definition of the centroid of the tetrahedron, we have

$$\begin{aligned}\overrightarrow{OG} &= \frac{1}{4}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}\right)\\ &= \frac{1}{2}\left(\frac{\overrightarrow{OA}+\overrightarrow{OB}}{2}\right)+\frac{1}{2}\left(\frac{\overrightarrow{OC}+\overrightarrow{OD}}{2}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OP}+\overrightarrow{OR}\right)\end{aligned}$$

and G is the midpoint of the segment PR.

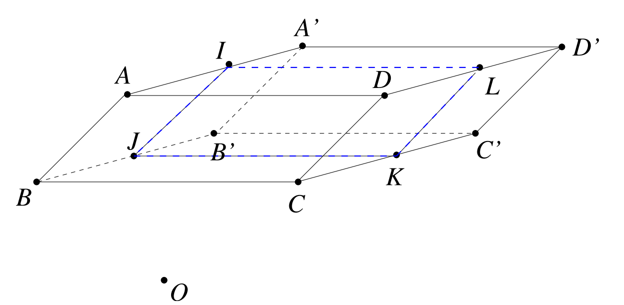
Similarly,

$$\begin{aligned}\overrightarrow{OG} &= \frac{1}{4}\left(\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}+\overrightarrow{OD}\right)\\ &= \frac{1}{2}\left(\frac{\overrightarrow{OB}+\overrightarrow{OC}}{2}\right)+\frac{1}{2}\left(\frac{\overrightarrow{OA}+\overrightarrow{OD}}{2}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OQ}+\overrightarrow{OS}\right)\end{aligned}$$

and G is the midpoint of the segment QS.

Question 10. Let $ABCD$ and $A'B'C'D'$ be two parallelograms, not necessarily coplanar, show that the midpoints I, J, K, L, of $[AA']$, $[BB']$, $[CC']$, $[DD']$, respectively, are the vertices of a parallelogram.

SOLUTION: In the figure, $ABCD$ and $A'B'C'D'$ are parallelograms, and O is an arbitrary point in the Euclidean point space $\mathcal{E}$.



Note first that

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{DC}, &\quad \text{and} \quad &\overrightarrow{AD} = \overrightarrow{BC},\\ \overrightarrow{A'B'} &= \overrightarrow{D'C'}, &\quad \text{and} \quad &\overrightarrow{A'D'} = \overrightarrow{B'C'}.\end{aligned}$$

Now

$$\begin{aligned}\overrightarrow{IJ} = \overrightarrow{OJ}-\overrightarrow{OI} &= \frac{1}{2}\left(\overrightarrow{OB}+\overrightarrow{OB'}\right)-\frac{1}{2}\left(\overrightarrow{OA}+\overrightarrow{OA'}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OB}-\overrightarrow{OA}\right)+\frac{1}{2}\left(\overrightarrow{OB'}-\overrightarrow{OA'}\right)\\ &= \frac{1}{2}\left(\overrightarrow{AB}+\overrightarrow{A'B'}\right).\end{aligned}$$

Also,

$$\begin{aligned}\overrightarrow{LK} = \overrightarrow{OK} - \overrightarrow{OL} &= \frac{1}{2}\left(\overrightarrow{OC} + \overrightarrow{OC'}\right) - \frac{1}{2}\left(\overrightarrow{OD} + \overrightarrow{OD'}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OD}\right) + \frac{1}{2}\left(\overrightarrow{OC'} - \overrightarrow{OD'}\right)\\ &= \frac{1}{2}\left(\overrightarrow{DC} + \overrightarrow{D'C'}\right).\end{aligned}$$

Since $\overrightarrow{AB} = \overrightarrow{DC}$ and $\overrightarrow{A'B'} = \overrightarrow{D'C'}$, then $\overrightarrow{IJ} = \overrightarrow{LK}$, so that IJ is parallel to LK and $|IJ| = |LK|$.

Similarly,

$$\begin{aligned}\overrightarrow{IL} = \overrightarrow{OL} - \overrightarrow{OI} &= \frac{1}{2}\left(\overrightarrow{OD} + \overrightarrow{OD'}\right) - \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OA'}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OD} - \overrightarrow{OA}\right) + \frac{1}{2}\left(\overrightarrow{OD'} - \overrightarrow{OA'}\right)\\ &= \frac{1}{2}\left(\overrightarrow{AD} + \overrightarrow{A'D'}\right),\end{aligned}$$

and

$$\begin{aligned}\overrightarrow{JK} = \overrightarrow{OK} - \overrightarrow{OJ} &= \frac{1}{2}\left(\overrightarrow{OC} + \overrightarrow{OC'}\right) - \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OB'}\right)\\ &= \frac{1}{2}\left(\overrightarrow{OC} - \overrightarrow{OB}\right) + \frac{1}{2}\left(\overrightarrow{OC'} - \overrightarrow{OB'}\right)\\ &= \frac{1}{2}\left(\overrightarrow{BC} + \overrightarrow{B'C'}\right).\end{aligned}$$

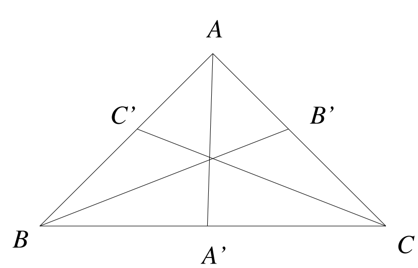
Since $\overrightarrow{AD} = \overrightarrow{BC}$ and $\overrightarrow{A'D'} = \overrightarrow{B'C'}$, then $\overrightarrow{IL} = \overrightarrow{JK}$, so that IL is parallel to JK and $|IL| = |JK|$.

Therefore, $IJKL$ is a parallelogram. Note that the above did not depend on the way the figure was drawn. The figure was drawn to make the process more transparent.

Question 11. Show that the angle bisectors of $\triangle ABC$ can be used to construct a new triangle if and only if $\triangle ABC$ is equilateral.

SOLUTION: Clearly, if $\triangle ABC$ is equilateral, then the angle bisectors are the perpendicular bisectors of the sides, and all have the same length. Therefore the angle bisectors can be used to construct a new triangle, and in fact, an equilateral triangle.

Conversely, let the angle bisectors of the angles at A, B, and C hit the opposite sides at the points A', B', and C', respectively, as in the figure, and let $a = BC$, $b = AC$, and $c = AB$.



Since the internal angle bisectors divide the opposite side (internally) in the ratio of the adjacent sides, then we have

$$\frac{BA'}{A'C} = \frac{AB}{AC} = \frac{c}{b},$$

$$\frac{AB'}{B'C} = \frac{AB}{BC} = \frac{c}{a},$$

$$\frac{AC'}{C'B} = \frac{AC}{BC} = \frac{b}{a},$$

and if the angle bisectors AA', BB', CC' can be used to form a new triangle, then we must have

$$\overrightarrow{AA'} + \overrightarrow{BB'} + \overrightarrow{CC'} = \overrightarrow{0}.$$

Since

$$\overrightarrow{AA'} = \overrightarrow{AB} + \overrightarrow{BA'} = \overrightarrow{AB} + \frac{c}{b+c}\overrightarrow{BC}$$

$$\overrightarrow{BB'} = \overrightarrow{BC} + \overrightarrow{CB'} = \overrightarrow{BC} + \frac{a}{a+c}\overrightarrow{CA}$$

$$\overrightarrow{CC'} = \overrightarrow{CA} + \overrightarrow{AC'} = \overrightarrow{CA} + \frac{b}{a+b}\overrightarrow{AB},$$

then

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} + \frac{c}{b+c}\overrightarrow{BC} + \frac{a}{a+c}\overrightarrow{CA} + \frac{b}{a+b}\overrightarrow{AB} = \overrightarrow{0},$$

and since

$$\overrightarrow{AB} + \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{0}, \qquad (\dagger)$$

this implies that

$$\frac{c}{b+c}\overrightarrow{BC} + \frac{a}{a+c}\overrightarrow{CA} + \frac{b}{a+b}\overrightarrow{AB} = \overrightarrow{0}. \qquad (\dagger\dagger)$$

Subtracting $\frac{c}{b+c}(\dagger)$ from $(\dagger\dagger)$, we have

$$\left(\frac{a}{a+c} - \frac{c}{b+c}\right)\overrightarrow{CA} + \left(\frac{b}{a+b} - \frac{c}{b+c}\right)\overrightarrow{AB} = \overrightarrow{0},$$

and since the set of vectors $\{\overrightarrow{AB}, \overrightarrow{CA}\}$ is linearly independent, then

$$\frac{a}{a+c} - \frac{c}{b+c} = 0 \qquad \text{and} \qquad \frac{b}{a+b} - \frac{c}{b+c} = 0,$$

that is, $c^2 = ab$ and $b^2 = ac$, so that $b^2 - c^2 = ac - ab = -a(b-c)$.

If $b \neq c$, this implies that $b + c = -a$, that is, $a + b + c = 0$, which is impossible. Therefore $b = c$, and then $ab = c^2 = b^2$ implies $a = b$, so that $\triangle ABC$ is equilateral.

Question 12. Let $X_1, X_2, \ldots, X_n$ be $n \geq 2$ points on a circle $\mathcal{C}$, and let G be their centroid. Denote by $Y_1, Y_2, \ldots, Y_n$ the second points of intersection of the lines $X_1G, X_2G, \ldots, X_nG$ with the circle, respectively.

(a) Show that

$$\frac{X_1G}{GY_1} + \frac{X_2G}{GY_2} + \cdots + \frac{X_nG}{GY_n} = n.$$

(b) Show that the set of points P inside the circle $\mathcal{C}$ that satisfy

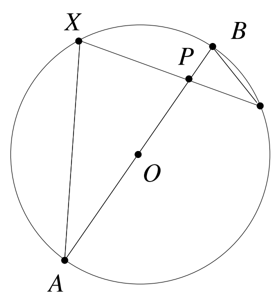
$$\frac{X_1P}{PY_1} + \frac{X_2P}{PY_2} + \cdots + \frac{X_nP}{PY_n} = n$$

is the circle with diameter OG, where O is the center of the circle $\mathcal{C}$.

Solution:

(a) Let r be the radius of the circle $\mathcal{C}$ and let O be its center. From the Power of a Point, we know that if P is any point inside $\mathcal{C}$, then for any two chords AB and XY intersecting at the point P, we have

$$AP \cdot PB = XP \cdot PY.$$



In the figure we apply this result to a chord XY and the diameter that both pass through P, then

$$XP \cdot PY = (r + OP)(r - OP) = r^2 - OP^2,$$

and therefore

$$\sum_{k=1}^{n} \frac{X_kP}{PY_k} = \sum_{k=1}^{n} \frac{X_kP^2}{X_kP \cdot PY_k} = \frac{1}{r^2 - OP^2} \sum_{k=1}^{n} X_kP^2. \qquad (*)$$

Since each of the points X_k are on the circle, then we have

$$\begin{aligned} X_kP^2 = \|\overrightarrow{X_kP}\|^2 &= \|\overrightarrow{OP} - \overrightarrow{OX_k}\|^2 \\ &= \|\overrightarrow{OP}\|^2 - 2\overrightarrow{OP} \cdot \overrightarrow{OX_k} + \|\overrightarrow{OX_k}\|^2 \\ &= OP^2 + r^2 - 2\overrightarrow{OP} \cdot \overrightarrow{OX_k}, \end{aligned}$$

where $\overrightarrow{OP} \cdot \overrightarrow{OX_k}$ is the Euclidean inner product of the two vectors $\overrightarrow{OP}$ and $\overrightarrow{OX_k}$.
Therefore,

$$\sum_{k=1}^{n} X_kP^2 = n(OP^2 + r^2) - 2\overrightarrow{OP} \cdot \sum_{k=1}^{n} \overrightarrow{OX_k},$$

and by definition of the centroid G, we have

$$\sum_{k=1}^{n} \overrightarrow{OX_k} = n\overrightarrow{OG},$$

so that

$$\sum_{k=1}^{n} X_k P^2 = n\left(OP^2 + r^2 - 2\overrightarrow{OP} \cdot \overrightarrow{OG}\right), \qquad (**)$$

Setting $P = G$, we have

$$\sum_{k=1}^{n} X_k G^2 = n(r^2 - OG^2),$$

so that

$$\sum_{k=1}^{n} \frac{X_k G}{GY_k} = \frac{1}{r^2 - OG^2} \sum_{k=1}^{n} X_k G^2 = \frac{n(r^2 - OG^2)}{r^2 - OG^2} = n.$$

(b) The argument above which lead to $(**)$ is valid for any point P inside the circle, and clearly the equation

$$\frac{X_1 P}{PY_1} + \frac{X_2 P}{PY_2} + \cdots + \frac{X_n P}{PY_n} = n$$

is equivalent to

$$\|\overrightarrow{OP}\|^2 = \overrightarrow{OP} \cdot \overrightarrow{OG},$$

which is equivalent to

$$\overrightarrow{OP} \cdot \overrightarrow{PG} = \overrightarrow{OP} \cdot \left(\overrightarrow{OG} - \overrightarrow{OP}\right) = \overrightarrow{OP} \cdot \overrightarrow{OG} - \|\overrightarrow{OP}\|^2 = 0,$$

that is, that the vectors $\overrightarrow{OP}$ and $\overrightarrow{PG}$ are perpendicular, and this condition defines the circle with diameter OG.