MATH 243 Winter 2008
Geometry II: Transformation Geometry

Solutions to Sample Quiz Problems

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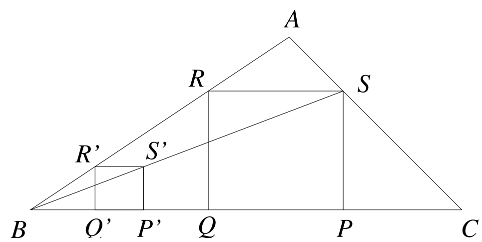
Question 1. Describe how to inscribe a square in a triangle.

SOLUTION: Given $\triangle ABC$, as in the figure,



from a point R' on AB, drop a perpendicular $R'Q'$ to BC, then complete the square $P'Q'R'S'$ with edge length $R'Q'$ and with $P'Q'$ in BC.

Next, draw the line BS' meeting AC at S, and drop the perpendicular SP to BC. Draw the line RS parallel to BC meeting AB at R, and note that $RS \perp SP$.

Finally, complete the parallelogram $PQRS$ with edges RS and SP, then $PQRS$ is the desired square.

To see this, note that since $QP \parallel RS$, we have $\angle SPQ = 180 - \angle PSR = 90$, so that PQ is in the side BC of $\triangle ABC$. Thus $PQRS$ is a rectangle, and we need only show that $PQRS$ is a square.

Since both $R'S'$ and RS are parallel to BC, then they are parallel to each other, and triangles $\triangle BRS$ and $\triangle BR'S'$ are similar, so that

$$\frac{RS}{R'S'} = \frac{BS}{BS'}.$$

Also, $S'P'$ and SP are parallel, so that triangles $\triangle BPS$ and $\triangle BP'S'$ are similar, so that

$$\frac{PS}{P'S'} = \frac{BS}{BS'}.$$

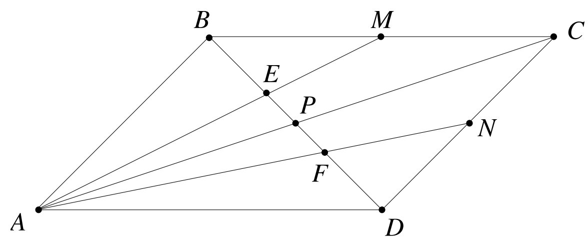
Therefore,

$$\frac{PS}{P'S'} = \frac{RS}{R'S'},$$

and since $P'S' = R'S'$, we get $PS = RS$, that is, the rectangle $PQRS$ is a square.

Question 2. Line seqments are drawn from a vertex of a parallelogram to the midpoints of the opposite sides. Use vectors to show that they trisect a diagonal.

SOLUTION: In the figure, $ABCD$ is a parallelogram, M is the midpoint of AB and N is the midpoint of CD. Since the diagonals of the parallelogram bisect each other then P is the midpoint of both AC and BD.



Now note that AM and BP are medians of $\triangle ABC$, so that E is the centroid of $\triangle ABC$, and therefore

$$\overrightarrow{BE} = \frac{2}{3}\overrightarrow{BP} = \frac{2}{3}\left(\frac{1}{2}\overrightarrow{BD}\right) = \frac{1}{3}\overrightarrow{BD}.$$

Similarly, AN and DP are medians of $\triangle ADC$, so that F is the centroid of $\triangle ADC$, and therefore

$$\overrightarrow{FD} = \frac{2}{3}\overrightarrow{PD} = \frac{2}{3}\left(\frac{1}{2}\overrightarrow{BD}\right) = \frac{1}{3}\overrightarrow{BD}.$$

Finally,

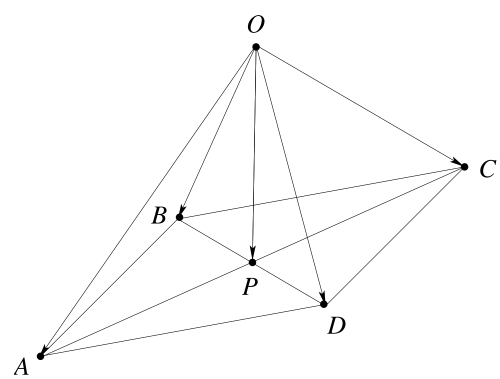
$$\overrightarrow{EF} = \overrightarrow{EB} + \overrightarrow{BD} + \overrightarrow{DF} = -\frac{1}{3}\overrightarrow{BD} + \overrightarrow{BD} - \frac{1}{3}\overrightarrow{BD} = \frac{1}{3}\overrightarrow{BD}.$$

Therefore the line segments AM and AN trisect the diagonal BD of the parallelogram $ABCD$.

Question 3. $ABCD$ is a parallelogram and P is the point of intersection of its diagonals. If O is an origin (not necessarily in the plane of the figure), show that

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4 \cdot \overrightarrow{OP}.$$

SOLUTION: In the figure,



note that since the diagonals of a parallelogram bisect each other, then

$$\overrightarrow{OP} = \overrightarrow{OA} + \frac{1}{2} \cdot \overrightarrow{AC} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right),$$

and

$$\overrightarrow{OP} = \overrightarrow{OB} + \frac{1}{2} \cdot \overrightarrow{BD} = \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OD}\right).$$

Adding these two equations, we get

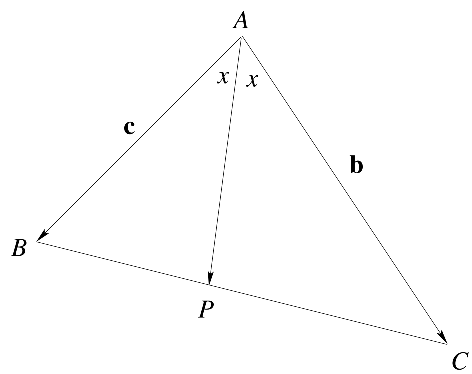
$$2 \cdot \overrightarrow{OP} = \frac{1}{2}\left(\overrightarrow{OA} + \overrightarrow{OC}\right) + \frac{1}{2}\left(\overrightarrow{OB} + \overrightarrow{OD}\right),$$

that is,

$$\overrightarrow{OA} + \overrightarrow{OB} + \overrightarrow{OC} + \overrightarrow{OD} = 4 \cdot \overrightarrow{OP}.$$

Question 4. Using vectors, show that the internal angle bisector of angle A of $\triangle ABC$ divides the side BC in the ratio $\dfrac{AB}{BC}$.

Solutions: Taking A as the origin, let $\mathbf{c} = \overrightarrow{AB}$ and $\mathbf{b} = \overrightarrow{AC}$, with $c = AB$ and $b = AC$, as in the figure,



then the internal angle bisector of angle A is on the line

$$\mathbf{r} = t\left(\widehat{\mathbf{b}} + \widehat{\mathbf{c}}\right) = t\left(\frac{\mathbf{b}}{b} + \frac{\mathbf{c}}{c}\right) = t\left(\frac{c\mathbf{b} + b\mathbf{c}}{bc}\right), \quad -\infty < t < \infty$$

and this line intersects the side $\overrightarrow{BC}$ when $t = \dfrac{bc}{b+c}$.

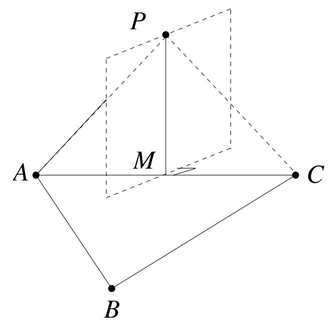
If P is the point of intersection of the internal angle bisector at A and the side $\overrightarrow{BC}$, then

$$\overrightarrow{AP} = \frac{c}{b+c}\mathbf{b} + \frac{b}{b+c}\mathbf{c},$$

so that P divides the side BC in the ratio $\dfrac{c}{b} = \dfrac{AB}{AC}$.

Question 5. If a point P is equidistant from the vertices of a right triangle and M is the midpoint of the hypotenuse, show using vectors that $\overrightarrow{PM}$ is perpendicular to the plane of the triangle.

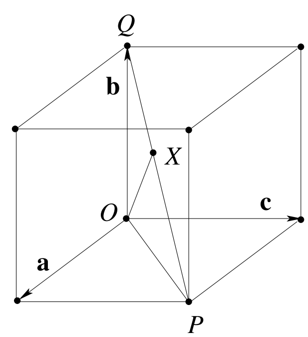
Solution: Let $\triangle ABC$ be a right triangle with the right angle at B, so that M is the midpoint of the side AC. Note that since P is equidistant from the vertices A and C then $\overrightarrow{PM}$ lies on the plane which is the perpendicular bisector of the side AC, and so $\overrightarrow{PM}$ is perpendicular to $\overrightarrow{AC}$.



Also, since P is equidistant from A and B, then P lies in the plane that is the perpendicular bisector of side AB, similarly P lies in the plane that is the perpendicular bisector of side BC. These three planes intersect in a line perpendicular to the plane of the triangle which passes through the circumcenter of $\triangle ABC$, and also through the point P. Since $\triangle ABC$ is a right triangle and M is the midpoint of the hypotenuse, its circumcenter is M, so that these three planes intersect in the line joining P and M. Therefore $\overrightarrow{PM}$ is perpendicular to the plane of the triangle.

Question 6. Using vectors, find the perpendicular distance of a corner of a unit cube from a diagonal not passing through the corner.

Solution: In the figure, we want to find the perpendicular distance from the corner O of the unit cube to the diagonal passing through the corners P and Q. We denote by $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ three mutually orthogonal unit vectors in the coordinate directions.



The vector from P to Q is given by

$$\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \mathbf{b} - (\mathbf{a} + \mathbf{c}) = \mathbf{b} - \mathbf{a} - \mathbf{c},$$

and a point X is on the line joining P and Q if and only if

$$\overrightarrow{OX} = \overrightarrow{OP} + \overrightarrow{PX} = \mathbf{a} + \mathbf{c} + t(\mathbf{b} - \mathbf{a} - \mathbf{c})$$

for some scalar $0 \le t \le 1$, that is, if and only if

$$\overrightarrow{OX} = t\mathbf{b} + (1-t)\mathbf{a} + (1-t)\mathbf{c}$$

for some $0 \le t \le 1$.

Now we want to find the point X of the line segment PQ such that $\overrightarrow{OX} \perp \overrightarrow{PQ}$, and taking the inner product, we have

$$(t\mathbf{b} + (1-t)\mathbf{a} + (1-t)\mathbf{c}) \cdot (\mathbf{b} - \mathbf{a} - \mathbf{c}) = 0$$

if and only if $t - (1-t) - (1-t) = 0$, that is, if and only if $3t = 2$, or $t = \dfrac{2}{3}$.

Therefore,

$$\overrightarrow{OX} = \frac{2}{3}\mathbf{b} + \frac{1}{3}\mathbf{a} + \frac{1}{3}\mathbf{c},$$

and

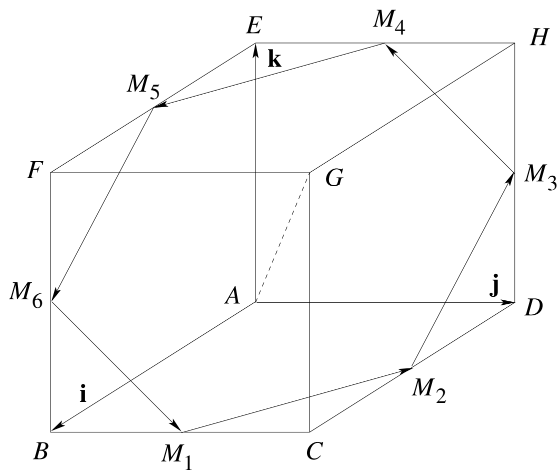
$$\|\overrightarrow{OX}\|^2 = \overrightarrow{OX} \cdot \overrightarrow{OX} = \frac{4}{9} + \frac{1}{9} + \frac{1}{9} = \frac{6}{9},$$

and the perpendicular distance from the corner O to the diagonal PQ is $\|\overrightarrow{OX}\| = \sqrt{\dfrac{2}{3}}$.

Note: A much simpler proof can be given by using similar right triangles in the plane containing the points O, P. and Q, but the problem said to use vectors.

Question 7. Using vectors, show that the midpoints of the six edges of a cube which do not meet a particular diagonal are coplanar.

Solution: In the figure, let M_i be the midpoints of the sides of the cube which do not intersect the diagonal AG, and let $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ be the vectors of length 2 in the direction of the coordinates axes as shown.



Let $\mathbf{a} = \mathbf{i} - \mathbf{j}$ and $\mathbf{b} = \mathbf{k} - \mathbf{j}$, then

$$\begin{aligned}
\overrightarrow{M_1M_2} &= -\mathbf{i} + \mathbf{j} = -\mathbf{a} \\
\overrightarrow{M_2M_3} &= -\mathbf{i} + \mathbf{k} = \mathbf{b} - \mathbf{a} \\
\overrightarrow{M_3M_4} &= \mathbf{k} - \mathbf{j} = \mathbf{b} \\
\overrightarrow{M_4M_5} &= -\mathbf{j} + \mathbf{i} = \mathbf{a} \\
\overrightarrow{M_5M_6} &= -\mathbf{k} + \mathbf{i} = \mathbf{a} - \mathbf{b} \\
\overrightarrow{M_6M_1} &= -\mathbf{k} + \mathbf{j} = -\mathbf{b},
\end{aligned}$$

so that all of the midpoints lie in the plane with system of coordinates $(M_1, \mathbf{a}, \mathbf{b})$.

Question 8. Let $\alpha : \mathcal{P} \longrightarrow \mathcal{P}$ be an isometry of the plane $\mathcal{P}$. Show that if P and Q are fixed points of α, then every point X on the line ℓ_{PQ} is a fixed point of α.

Solution: If $\alpha(P) = P$ and $\alpha(Q) = Q$, then $\alpha(P)$ and $\alpha(Q)$ are on the line $\alpha(\ell_{PQ})$, that is, P and Q are on the line $\alpha(\ell_{PQ})$, and since α maps lines onto lines, then $\alpha(\ell_{PQ}) = \ell_{PQ}$.

Therefore, if X is a point on the line ℓ_{PQ}, then $\alpha(X)$ is on the line ℓ_{PQ}, and since α is distance preserving, then

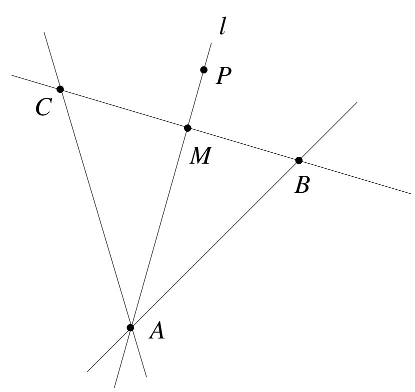
$$d(\alpha(X), P) = d(\alpha(X), \alpha(P)) = d(X, P) \qquad \text{and} \qquad d(\alpha(X), Q) = d(\alpha(X), \alpha(Q)) = d(X, Q).$$

Since P, Q, X, $\alpha(X)$ are all on the same line ℓ_{PQ} and the isometry α preserves betweenness, then $\alpha(X) = X$.

Question 9. Let $\alpha : \mathcal{P} \longrightarrow \mathcal{P}$ be an isometry of the plane $\mathcal{P}$. Show that if α has three fixed points which are not collinear, then $\alpha = \iota$ (the identity).

Solution: Suppose that A, B, and C are three noncollinear fixed points of the isometry α, then every point of the lines ℓ_{AB}, ℓ_{BC}, and ℓ_{AC} is a fixed point.

Let P be any point different from A, and let M be the intersection of ℓ_{BC} with $\ell = \ell_{AP}$, as in the figure.



Since B and C are fixed points, then every point of ℓ_{BC} is also a fixed point, therefore M is a fixed point of α, and therefore every point of ℓ is also a fixed point of α, thus P is a fixed point of α.

The same argument applies to points of the line through A parallel to ℓ_{BC} if we replace the line ℓ_{AP} by ℓ_{BP}. Therefore, $\alpha(P) = P$ for all $p \in \mathcal{P}$, that is, $\alpha = \iota$.

Question 10. Let α and β be isometries, and let A, B, C three noncollinear points for which

$$\alpha(A) = \beta(A), \qquad \alpha(B) = \beta(B), \qquad \text{and} \qquad \alpha(C) = \beta(C).$$

Show that $\alpha = \beta$.

Solution: If we let $\gamma = \alpha\beta^{-1}$, then γ is an isometry and

$$\gamma(A) = A, \qquad \gamma(B) = B, \qquad \text{and} \qquad \gamma(C) = C,$$

and from the previous problem, since A, B, and C are noncollinear, then $\gamma = \alpha\beta^{-1} = \iota$, that is, $\alpha = \beta$.