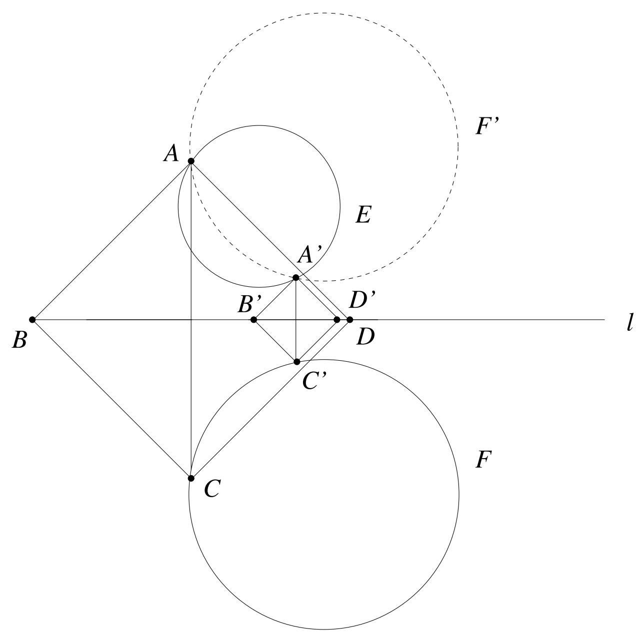
Math243

Solutions to Midterm Examination

Question 1. [**10 points**] Given two circles E and F separated by a line ℓ, find all squares $ABCD$ with vertex A on E, the opposite vertex C on F, and the remaining vertices on ℓ.

SOLUTION: We apply the transformation σ_ℓ to the circle F, and the points where the image F' intersects E give us the point (or points) A of the desired square. From A we drop a perpendicular to the line ℓ to obtain the point (or points) C on the circle F.

Now that we have the points A and C on the diagonal, we can construct the square $ABCD$ with points B and D on the line ℓ.



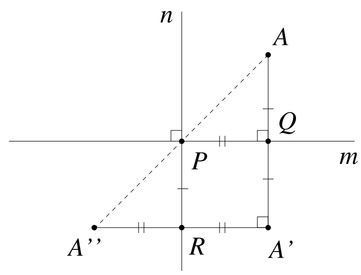
As shown in the figure, there are two such squares that can be constructed, since there are two intersection points of the circles E and F'.

Question 2. [10 points] Show that if m and n are perpendicular lines that intersect at a point $P \in \mathcal{P}$, then

$$\sigma_n \, \sigma_m = \sigma_P.$$

Solution: Let $A \in \mathcal{P}$ be arbitrary, we will show that $\sigma_n(\sigma_m(A)) = \sigma_P(A)$.

We let $A' = \sigma_m(A)$, and $A'' = \sigma_n(A')$, as in the figure.



Since $\triangle AQP$ and $\triangle PRA''$ are congruent by the SAS congruency theorem, then $\angle APQ = \angle PA''R$, so that A, P, and A'' are collinear. Also, $AP = A''P$, so that $A'' = \sigma_P(A)$.

Therefore

$$\sigma_P(A) = A'' = \sigma_n(A') = \sigma_n(\sigma_m(A))$$

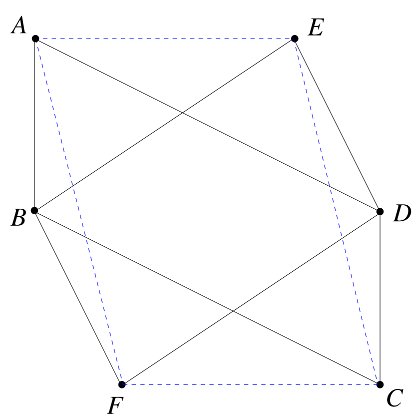
for all $A \in \mathcal{P}$, so that $\sigma_n \, \sigma_m = \sigma_P$.

Question 3. [10 points] Prove using halfturns, that if $ABCD$ and $EBFD$ are parallelograms, then $EAFC$ is also a parallelogram.

Solution: Note that

$$\sigma_A \sigma_B \sigma_C \sigma_D = \iota \qquad \text{and} \qquad \sigma_D \sigma_F \sigma_B \sigma_E = \iota$$

since $ABCD$ and $EBFD$ are parallelograms.



Therefore,

$$\sigma_A \sigma_B \sigma_C \sigma_D \sigma_D \sigma_F \sigma_B \sigma_E = \iota^2 = \iota,$$

and since $\sigma_D^2 = \iota$, then

$$\sigma_A \sigma_B \sigma_C \sigma_F \sigma_B \sigma_E = \iota.$$

Since $\sigma_B\sigma_C\sigma_F = \sigma_F\sigma_C\sigma_B$, then

$$\sigma_A\sigma_F\sigma_C\sigma_B\sigma_B\sigma_E = \iota,$$

and since $\sigma_B^2 = \iota$, then

$$\sigma_A\sigma_F\sigma_C\sigma_E = \iota,$$

so that $EAFC$ is a parallelogram.

Question 4. [10 points] Given triangles $\triangle OAB$ and $\triangle OA'B'$, where

$$O(0,0), \quad A(0,1), \quad B(1,1), \quad A'(1,0), \quad B'(1,-1).$$

Find the equations of two lines m and n such that the product of reflections in these lines maps $\triangle OAB$ onto $\triangle OA'B'$. What is the product of these two reflections?

Solution: The first reflection is about the perpendicular bisector of the line segment joining A and A'. We reflect about the line $m:\ y = x$, so the first reflection σ_m has equations

$$(x'', y'') = \sigma_m(x, y) = (y, x),$$

so that

$$\begin{aligned} \sigma_m(O) &= \sigma_m(0,0) = (0,0) = O, \\ \sigma_m(A) &= \sigma_m(0,1) = (1,0) = A', \\ \sigma_m(B) &= \sigma_m(1,1) = (1,1) = B'', \end{aligned}$$

and $\triangle OAB \equiv \triangle OA'B''$.

The second reflection is about the perpendicular bisector of the line segment joining B and B''. We reflect about the line $n:\ y = 0$, so the second reflection σ_n has equations

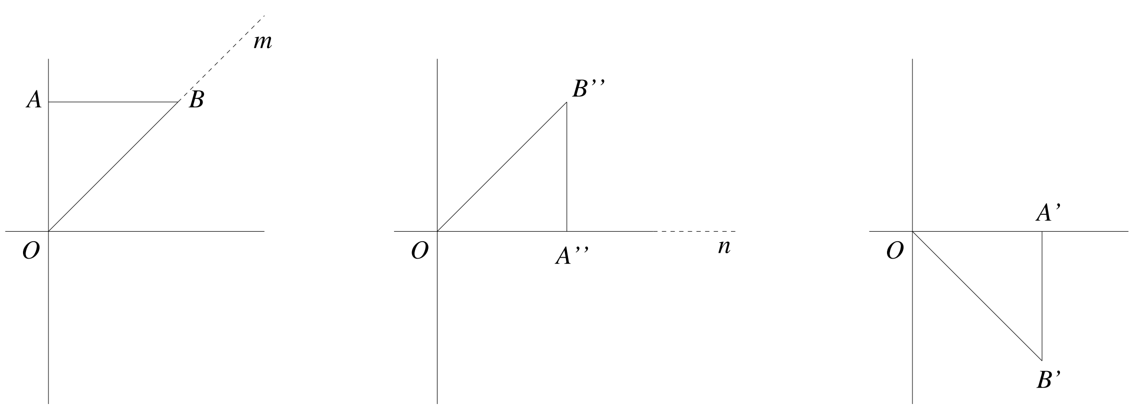
$$(x'', y'') = \sigma_n(x, y) = (x, -y),$$

so that

$$\begin{aligned} \sigma_n(O) &= \sigma_n(0,0) = (0,0) = O, \\ \sigma_n(A') &= \sigma_n(1,0) = (1,0) = A', \\ \sigma_n(B'') &= \sigma_n(1,1) = (1,-1) = B', \end{aligned}$$

and $\triangle OA'B'' \equiv \triangle OA'B'$.

Therefore the product of reflections $\sigma_n\sigma_m$ maps $\triangle OAB$ onto $\triangle OA'B'$, as shown below.



The isometry above is just a rotation about the point O by an angle of $-90°$, that is,

$$\sigma_n\sigma_m = \rho_{O,-90} \,.$$