MATH 243 Winter 2008
Geometry II: Transformation Geometry
Problem Set 2
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Question 1. Which of the mappings defined on the Cartesian plane $\mathcal{P}$ by the equations below are transformations?

(a) $\alpha(x, y) = (x^3, y^3)$ (b) $\beta(x, y) = (x, y^2)$

(c) $\gamma(x, y) = (3y, x + 2)$ (d) $\delta(x, y) = (x + y + 3, 2x + 2y - 1)$

Question 2. Let $\alpha(x, y) = (x + 1, y + 2x)$ and $\beta = (x + y - 1, y)$ be two mappings defined on the Cartesian plane $\mathcal{P}$.

(a) Show that α and β are transformations of $\mathcal{P}$.

(b) Find $\alpha\,\beta$ and $\beta\,\alpha$.

(c) Find α^{-1} and β^{-1}.

Question 3.

(a) Find the image of the line $2x + 3y = 1$ under the affine transformation

$$\alpha(x, y) = (x + y + 1, x - y + 2).$$

(b) Find the fixed points of α.

Question 4.

(a) Prove that any affine transformation is a collineation.

(b) Show that $\alpha(x, y) = (2x^3 + 1, y^3)$ is a transformation of the plane but is not a collineation.

Question 5. Let α and β be two involutive transformations of the Cartesian plane $\mathcal{P}$.

(a) Prove that $\alpha\,\beta$ is involutive if and only if $\alpha\,\beta = \beta\,\alpha$.

(b) Assume that α, β, ι are distinct transformations such that

$$\alpha\,\beta = \beta\,\alpha = \gamma.$$

Let $\Gamma = \{\iota, \alpha, \beta, \gamma\}$. Prove that Γ is a commutative subgroup of $\mathcal{G}$, the group of all transformations on the plane $\mathcal{P}$ (construct the multiplication table).

Question 6. Let $\alpha(x,y) = (ax+by, cx+dy)$ be an affine transformation of $\mathcal{P}$. Prove that α is an involution if and only if

$$\begin{aligned} a^2 + bc &= 1 \\ ab + bd &= 0 \\ ac + cd &= 0 \\ bc + d^2 &= 1. \end{aligned}$$

Note: The matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is called the matrix of the transformation α. The conditions above say that α is an involution if and only if $A^2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

Question 7. Let α be an isometry of $\mathcal{P}$ which admits an invariant line ℓ (that is, $\alpha(\ell) = \ell$) and a fixed point $P \in \mathcal{P}$. Prove that there is a point $Q \in \ell$ such that $\alpha(Q) = Q$ and a line ℓ' through P such that $\alpha(\ell') = \ell'$.

Question 8. If a circle is invariant under the isometry α then its center is a fixed point of α.

Question 9. Let $\alpha \neq \iota$ be an involutive isometry, show that α has at least one fixed point.

Question 10. Let α be an isometry of $\mathcal{P}$ and let ℓ be the perpendicular bisector of the segment $[AB]$. Prove that $\alpha(\ell)$ is the perpendicular bisector of the segment $[\alpha(A)\,\alpha(B)]$.