MATH 243 Winter 2008
Geometry II: Transformation Geometry
Problem Set 1
Completion Date: Monday January 21, 2008

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Question 1. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors, both parallel to a line ℓ.

(a) Show that $\mathbf{u}+\mathbf{v}$ is parallel to ℓ.

(b) Show that $k\mathbf{u}$ is parallel to ℓ for each $k \in \mathbb{R}$, $k \neq 0$.

Question 2. Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors, both parallel to a plane Π.

(a) Show that $\mathbf{u}+\mathbf{v}$ is parallel to Π.

(b) Show that $k\mathbf{u}$ is parallel to Π for each $k \in \mathbb{R}$, $k \neq 0$.

Question 3. Given an arbitrary point O, let A', B', C', be the midpoints of the sides BC, AC, and AB of $\triangle ABC$, show that

$$\overrightarrow{OA}+\overrightarrow{OB}+\overrightarrow{OC}=\overrightarrow{OA'}+\overrightarrow{OB'}+\overrightarrow{OC'}.$$

Question 4. Given $\triangle A\,B\,C$ and $\triangle A'\,B'\,C'$, let G and G' be their centroids, respectively. Show that

$$\overrightarrow{GG'}=\frac{1}{3}\left(\overrightarrow{AA'}+\overrightarrow{BB'}+\overrightarrow{CC'}\right).$$

Question 5. Show using vectors that the segment joining the midpoints of the nonparallel sides of a trapezoid is parallel to either base and congruent to half the sum of the bases.

Question 6. If the point P lies in the plane of $\triangle ABC$, but is distinct from the vertices of the triangle, and the parallelograms $PBA'C$, $PCB'A$, $PAC'B$ are completed, show that the segments $[AA']$, $[BB']$, and $[CC']$ bisect each other.

Question 7. Suppose that the points A', B', C' lie on the sides $[BC]$, $[CA]$, and $[AB]$ of $\triangle ABC$, respectively, and

$$\frac{A'B}{A'C}=\frac{B'C}{B'A}=\frac{C'A}{C'B}.$$

(a) Show that the centroids of $\triangle A'B'C'$ and $\triangle ABC$ coincide.

(b) If the parallelograms $AC'C''C$ and $AB'B''B$ are completed, show that $B'C''$ and $C'B''$ are parallel to the median of $\triangle ABC$ through A.

Question 8. If A_1, A_2, A_3, A_4, A_5, A_6 are the midpoints of consecutive sides of a hexagon, show that $\triangle A_1A_3A_5$ and $\triangle A_2A_4A_6$ have the same centroid.

Question 9. If P, Q, R, S are the midpoints of the edges AB, BC, CD, DA, respectively, of a tetrahedron $ABCD$, show that PR and QS bisect each other at the centroid of the tetrahedron.

Question 10. Let $ABCD$ and $A'B'C'D'$ be two parallelograms, not necessarily coplanar, show that the midpoints I, J, K, L, of $[AA']$, $[BB']$, $[CC']$, $[DD']$, respectively, are the vertices of a parallelogram.

Question 11. Show that the angle bisectors of $\triangle ABC$ can be used to construct a new triangle if and only if $\triangle ABC$ is equilateral.

Question 12. Let X_1, X_2, $\ldots$, X_n be $n \geq 2$ points on a circle $\mathcal{C}$, and let G be their centroid. Denote by Y_1, Y_2, $\ldots$, Y_n the second points of intersection of the lines X_1G, X_2G, $\ldots$, X_nG with the circle, respectively.

(a) Show that

$$\frac{X_1G}{GY_1} + \frac{X_2G}{GY_2} + \cdots + \frac{X_nG}{GY_n} = n.$$

(b) Show that the set of points P inside the circle $\mathcal{C}$ that satisfy

$$\frac{X_1P}{PY_1} + \frac{X_2P}{PY_2} + \cdots + \frac{X_nP}{PY_n} = n$$

is the circle with diameter OG, where O is the center of the circle $\mathcal{C}$.