math 228

Solutions to Midterm Examination

Instructor: I. E. Leonard **Time: 70 Minutes**

1. Define the binary operation $* : \mathbb{Z} \times \mathbb{Z} \longrightarrow \mathbb{Z}$ on the set of integers $\mathbb{Z}$ by

$$a * b = a + b - 1$$

for $a,\, b \in \mathbb{Z}$.

(a) Show that $*$ is commutative.

(b) Show that $*$ is associative.

(c) Find an identity element e for the binary operation $*$, that is, an element $e \in \mathbb{Z}$ such that $x * e = x = e * x$ for all $x \in \mathbb{Z}$.

(d) Show that each $a \in \mathbb{Z}$ has an inverse with respect to the binary operation $*$, that is, show that for each $a \in \mathbb{Z}$ there exists a $b \in \mathbb{Z}$ such that $a * b = e = b * a$.

Solution:

(a) $a * b = a + b - 1 = b + a - 1 = b * a$ for all $a, b \in \mathbb{Z}$.

(b) $a * (b * c) = a + (b + c - 1) - 1 = (a + b - 1) + c - 1 = (a * b) * c$ for all $a, b, c \in \mathbb{Z}$.

(c) If $x * e = x + e - 1 = x$, then $e - 1 = 0$, so the identity element for the binary operation $*$ is $e = 1$.

(d) If $a, b \in \mathbb{Z}$ and $a * b = e$, then $a + b - 1 = 1$, so that $a + b = 2$. The inverse of $a \in \mathbb{Z}$ with respect to the binary operation $*$ is $b = 2 - a$.

2. Let $A = \{a, b, c, d\}$ and define the binary operations addition $+$ and multiplication $\cdot$ by the tables

$+$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	b	c	d
c	a	c	c	a
d	a	d	a	d

Use these tables to determine which of the following are true:

(a) $(a + b) + c = a + (b + c)$

(b) $(b + c) + d = b + (c + d)$

(c) $c \cdot (b + d) = c \cdot b + c \cdot d$

(d) $(c + b) \cdot d = c \cdot d + b \cdot d$

(e) $(d \cdot b) \cdot c = d \cdot (b \cdot c)$

SOLUTION:

(a) From the tables, we have

$$(a+b)+c = b+c = d \quad \text{and} \quad a+(b+c) = a+d = d$$

so that $(a+b)+c = a+(b+c)$ is true.

(b) From the tables, we have

$$(b+c)+d = d+d = a \quad \text{and} \quad b+(c+d) = b+b = a$$

so that $(b+c)+d = b+(c+d)$ is true.

(c) From the tables, we have

$$c\cdot(b+d) = c\cdot c = c \quad \text{and} \quad c\cdot b + c\cdot d = c+a = c$$

so that $c\cdot(b+d) = c\cdot b + c\cdot d$ is true.

(d) From the tables, we have

$$(c+b)\cdot d = d\cdot d = d \quad \text{and} \quad c\cdot d + b\cdot d = a+d = d$$

so that $(c+b)\cdot d = c\cdot d + b\cdot d$ is true.

(e) From the tables, we have

$$(d\cdot b)\cdot c = d\cdot c = a \quad \text{and} \quad d\cdot(b\cdot c) = d\cdot c = a$$

so that $(d\cdot b)\cdot c = d\cdot(b\cdot c)$ is true.

3. The following is the addition table and part of the multiplication table for a ring with three elements.

$+$	a	b	c
a	a	b	c
b	b	c	a
c	c	a	b

$\cdot$	a	b	c
a	a	a	a
b	a	b	
c	a		

(a) What is the additive identity?

(b) What is the additive inverse of b?

(c) What is the additive inverse of c?

(d) Use the distributive laws to fill in the rest of the multiplication table.

(e) Is this a commutative ring? Does it have a multiplicative identity?

SOLUTION:

(a) From the addition table we have

$$\begin{aligned} a+a &= a = a+a \\ a+b &= b = b+a \\ a+c &= c = c+a \end{aligned}$$

and the additive identity is a.

(b) Now that we know the additive identity is a, from the addition table we have

$$b+c = c+b = a$$

and the additive inverse of b is c, that is, $-b = c$.

(c) Since $b + c = c + b = a$, then the additive inverse of c is b, that is, $-c = b$.

(d) From the addition and multiplication tables we have

$$b + b \cdot c = b \cdot b + b \cdot c = b \cdot (b + c) = b \cdot a = a,$$

that is, the additive inverse of b is $b \cdot c$, and since additive inverses are unique, then $b \cdot c = c$.
Similarly,

$$b + c \cdot b = b \cdot b + c \cdot b = (b + c) \cdot b = a \cdot b = a,$$

that is, $c \cdot b$ is the additive inverse of b, and by uniqueness of additive inverses, $c \cdot b = c$.
Finally, from the addition and multiplication tables, we have

$$c \cdot c = c \cdot (b + b) = c \cdot b + c \cdot b = c + c = b.$$

The completed multiplication table is given below.

$\cdot$	a	b	c
a	a	a	a
b	a	b	c
c	a	c	b

(e) Since the multiplication table is symmetric about the main diagonal, then $x \cdot y = y \cdot x$ for all $x, y \in \{a, b, c\}$, and the ring is commutative.
Also, from the second row and the second column in the multiplication table we see that

$$b \cdot a = a = a \cdot b, \quad b \cdot b = b = b \cdot b, \quad b \cdot c = c = c \cdot b$$

so that $b \cdot x = x = x \cdot b$ for all $x \in \{a, b, c\}$, and the multiplicative identity is b.

4. Let

$$a_n = \frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots + \frac{1}{n(n+1)}$$

for all $n \geq 1$.

(a) Compute a_1, a_2, a_3, a_4, a_5.

(b) Prove that $a_n = \dfrac{n}{n+1}$ for all $n \geq 1$ using the principle of mathematical induction.

Solution:

(a) We have

$$a_1 = \frac{1}{1 \cdot 2} = \frac{1}{2}$$

$$a_2 = a_1 + \frac{1}{2 \cdot 3} = \frac{1}{2} + \frac{1}{6} = \frac{4}{6} = \frac{2}{3}$$

$$a_3 = a_2 + \frac{1}{3 \cdot 4} = \frac{2}{3} + \frac{1}{12} = \frac{9}{12} = \frac{3}{4}$$

$$a_4 = a_3 + \frac{1}{4 \cdot 5} = \frac{3}{4} + \frac{1}{20} = \frac{16}{20} = \frac{4}{5}$$

$$a_5 = a_4 + \frac{1}{5 \cdot 6} = \frac{4}{5} + \frac{1}{30} = \frac{25}{30} = \frac{5}{6}$$

(b) We will use the principle of mathematical induction to show that

$$a_n = \frac{n}{n+1}$$

for all $n \geq 1$.

The base case for $n = 1$ is true since

$$a_1 = \frac{1}{1(1+1)} = \frac{1}{1 \cdot 2} = \frac{1}{2} = \frac{1}{1+1}.$$

Assume that the result is true for some $n \geq 1$, so that $a_n = \dfrac{n}{n+1}$ for some $n \geq 1$, then

$$a_{n+1} = a_n + \frac{1}{(n+1)(n+2)} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)}$$

$$= \frac{1}{(n+1)(n+2)}\left[n(n+2)+1\right] = \frac{1}{(n+1)(n+2)}\left[n^2+2n+1\right]$$

$$= \frac{(n+1)^2}{(n+1)(n+2)} = \frac{n+1}{n+2}$$

and the result is true for $n + 1$ also. Therefore, by the principle of mathematical induction, the result is true for all $n \geq 1$.

5. (a) Use the Euclidean algorithm to find $(629, 2431)$.
   (b) Find the integer solutions to the equation $629 \cdot x + 2431 \cdot y = 102$.

SOLUTION:

(a) Applying the Euclidean algorithm to $a = 629$ and $b = 2431$ we have

$$\begin{aligned} 629 &= 0 \cdot 2431 + 629 \\ 2413 &= 3 \cdot 629 + 544 \\ 629 &= 1 \cdot 544 + 85 \\ 544 &= 6 \cdot 85 + 34 \\ 85 &= 2 \cdot 34 + 17 \end{aligned}$$

so that $(629, 2431) = 17$.

Note that if we had started with $a = 2431$ and $b = 629$ we would have obtained the same result, but in one less step.

(b) Working from the bottom up in the Euclidean algorithm, we can write $17 = (629, 2431)$ as a linear combination of 629 and 2431 as follows:

$$\begin{aligned} 17 &= 85 - 2 \cdot 34 \\ &= 85 - 2(544 - 6 \cdot 85) = 13 \cdot 85 - 2 \cdot 544 \\ &= 13(629 - 1 \cdot 544) - 2 \cdot 544 = 13 \cdot 629 - 15 \cdot 544 \\ &= 13 \cdot 629 - 15(2431 - 3 \cdot 629) = 58 \cdot 629 - 15 \cdot 2431 \end{aligned}$$

and therefore $17 = 58 \cdot 629 - 15 \cdot 2431$.

Now, since $102 = 6 \cdot 17$, then $17 \mid 102$ so that the equation

$$629 \cdot x + 2431 \cdot y = 102$$

has a solution, and we have

$$(6 \cdot 58) \cdot 629 - (6 \cdot 15) \cdot 2431 = 6 \cdot 17 = 102,$$

that is, $348 \cdot 629 - 90 \cdot 2431 = 102$ and an integer solution to the equation is $x_0 = 348$, $y_0 = -90$. Now note that for any $k \in \mathbb{Z}$,

$$\begin{aligned} x &= x_0 + \tfrac{2413}{17} \cdot k \\ y &= y_0 - \tfrac{629}{17} \cdot k \end{aligned}$$

is also a solution to the equation, and so the solutions are given by

$$\begin{aligned} x &= 348 + 143 \cdot k \\ y &= -90 - 37 \cdot k \end{aligned}$$

where $k \in \mathbb{Z}$.