math 228

Assignment 4, due Monday June 11, 2007

Question 1. [Exercises 3.3, # 2]

Let the ring $R = \{0,\ e,\ b,\ c\}$ with addition and multiplication defined by the following tables. Use tables to show that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is isomorphic to the ring R.

$+$	0	e	b	c
0	0	e	b	c
e	e	0	c	b
b	b	c	0	e
c	c	b	e	0

$\cdot$	0	e	b	c
0	0	0	0	0
e	0	e	b	c
b	0	b	b	0
c	0	c	0	c

Question 2. [Exercises 3.3, # 7]

Let $\mathbb{Q}(\sqrt{2}) = \{r + s\sqrt{2} \,|\, r, s \in \mathbb{Q}\}$.

(a) Show that $\mathbb{Q}(\sqrt{2})$ is a subfield of $\mathbb{R}$.

(b) Prove that the function $f : \mathbb{Q}(\sqrt{2}) \longrightarrow \mathbb{Q}(\sqrt{2})$ given by

$$f(a + b\sqrt{2}) = a - b\sqrt{2}, \quad a, b \in \mathbb{Q}$$

is an isomorphism.

Question 3. [Exercises 4.1, # 3].

(a) List all polynomials of degree 3 in $\mathbb{Z}_2[\,x\,]$.

(b) List all polynomials of degree less than 3 in $\mathbb{Z}_3[\,x\,]$.

Question 4. [Exercises 4.1, # 10].

If $f(x), g(x) \in R[\,x\,]$ and $f(x) + g(x) \neq 0_R$, show that

$$\deg[f(x) + g(x)] \leq \max\{\deg f(x), \deg g(x)\}.$$

Question 5. [Exercises 4.2, # 4].

(a) Let $f(x), g(x) \in F[\,x\,]$, where F is a field. If $f(x) \mid g(x)$ and $g(x) \mid f(x)$, show that $f(x) = cg(x)$ for some nonzero $c \in F$.

(b) If $f(x)$ and $g(x)$ in part (a) are monic, show that $f(x) = g(x)$.

Question 6. [Exercises 4.2, # 10].

Find the gcd of $x + a + b$ and $x^3 - 3abx + a^3 + b^3$ in $\mathbb{Q}[\,x\,]$.

Question 7. [Exercises 4.3, # 3].

List all associates of

(a) $x^2 + x + 1$ in $\mathbb{Z}_5[\,x\,]$. (b) $3x + 2$ in $\mathbb{Z}_7[\,x\,]$.

Question 8. [Exercises 4.3, # 12].

Express $x^4 - 4$ as a product of irreducibles in $\mathbb{Q}[\,x\,]$, in $\mathbb{R}[\,x\,]$, and in $\mathbb{C}[\,x\,]$.

Question 9. [Exercises 4.4, # 10].

Find a prime $p > 5$ such that $x^2 + 1$ is reducible in $\mathbb{Z}_p[\,x\,]$.

Question 10. [Exercises 4.5, # 3].

If a monic polynomial with integer coefficients has a root in $\mathbb{Q}$, show that this root must be an integer.

Question 11. [Exercises 4.6, #6].

Let $f(x) = ax^2 + bx + c \in \mathbb{R}[\,x\,]$ with $a \neq 0$. Prove that the roots of $f(x)$ in $\mathbb{C}$ are

$$\frac{-b + \sqrt{b^2 - 4ac}}{2a} \qquad \text{and} \qquad \frac{-b - \sqrt{b^2 - 4ac}}{2a}.$$

Hint: Show that $ax^2 + bx + c = 0$ is equivalent to $x^2 + (b/a)x = -c/a$; then complete the square to find x.