math 228

Assignment 1, due Friday May 11, 2007

Question 1. [Exercises 1.1, # 6]

Use the division algorithm to prove that every odd integer is either of the form $4k + 1$ or of the form $4k + 3$ for some integer k.

Question 2. [Exercises 1.1, # 8]

(a) Divide 5^2, 7^2, 11^2, 15^2, and 27^2 by 8 and note the remainder in each case.

(b) Make a conjecture about the remainder when the square of an odd integer is divided by 8.

(c) Prove your conjecture.

Question 3. [Exercises 1.2, # 8].

If $r \in \mathbb{Z}$ and r is a nonzero solution of $x^2 + ax + b = 0$ (where a, $b \in \mathbb{Z}$), prove that $r \mid b$.

Question 4. [Exercises 1.2, # 10].

Prove that $(n, n + 1) = 1$ for every positive integer n.

Question 5. [Exercises 1.2, # 16].

If $(a, b) = d$, prove that $\left(\frac{a}{d}, \frac{b}{d}\right) = 1$.

Question 6. [Exercises 1.2, # 26].

Let a, b, $c \in \mathbb{Z}$. Prove that the equation $ax + by = c$ has integer solutions if and only if $(a, b) \mid c$.

Question 7. [Exercises 1.2, # 32].

Prove that a positive integer is divisible by 3 if and only if the sum of its digits is divisible by 3.

[*Hint*: $10^3 = 999 + 1$ and similarly for other powers of 10.]

Question 8. [Exercises 1.3, # 2].

Let p be an integer other than 0, ± 1. Prove that p is prime if and only if for each $a \in \mathbb{Z}$ either $(a, p) = 1$ or $p \mid a$.

Question 9. [Exercises 1.3, # 8].

Prove that $(a, b) = 1$ if and only if there is no prime p such that $p \mid a$ and $p \mid b$.

Question 10. [Exercises 1.3, # 12].

(a) If $3 \mid (a^2 + b^2)$, prove that $3 \mid a$ and $3 \mid b$.
[*Hint*: If $3 \nmid a$, then $a = 3k + 1$ or $a = 3k + 2$.]

(b) If $5 \mid (a^2 + b^2 + c^2)$, prove that $5 \mid a$ or $5 \mid b$ or $5 \mid c$.

Question 11. [Exercises 1.3, # 20].

(a) Prove that there are no nonzero integers a, b such that $a^2 = 2b^2$.
[*Hint*: Use the Fundamental Theorem of Arithmetic or Theorem 1.8.]

(b) Prove that $\sqrt{2}$ is irrational.
[*Hint*: Use proof by contradiction (Appendix A). Assume that $\sqrt{2} = a/b$ (with a, $b \in \mathbb{Z}$) and use part (a) to reach a contradiction.]