MATH 225 Summer 2005
Linear Algebra II
Assignment 5
Due: Wednesday August 10, 2005

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Question 1. [p 436. #14]

Let T be a one-to-one linear transformation from a vector space V into $\mathbb{R}^n$. Show that for $\mathbf{u}$, $\mathbf{v}$ in V, the formula

$$< \mathbf{u}, \mathbf{v} >= T(\mathbf{u}) \cdot T(\mathbf{v})$$

defines an inner product on V.

Question 2. [p 454. #22]

Let $A = \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$.

Diagonalize A by finding an orthogonal matrix P and a diagonal matrix D such that $A = PDP^T$.

Question 3. [p 455. #28]

Show that if A is an $n \times n$ symmetric matrix, then

$$(A\mathbf{x}) \cdot \mathbf{y} = \mathbf{x} \cdot (A\mathbf{y})$$

for all $\mathbf{x}$, $\mathbf{y}$ in $\mathbb{R}^n$.

Question 4. [p 455. #34]

Let $A = \begin{pmatrix} 3 & -2 & 4 \\ -2 & 6 & 2 \\ 4 & 2 & 3 \end{pmatrix}$.

(a) Orthogonally diagonalize A.

(b) Construct a spectral decomposition of A.

Question 5. [p 462. #6]

Find the matrix of the quadratic form. Assume $\mathbf{x}$ is in $\mathbb{R}^3$.

(a) $5x_1^2 - x_2^2 + 7x_3^2 + 5x_1x_2 - 3x_1x_3$

(b) $x_3^2 - 4x_1x_2 + 4x_2x_3$

Question 6. [p 463. #26]

Show that if an $n \times n$ matrix A is positive definite, then there exists a positive definite matrix B such that $A = B^T B$.

Hint: Write $A = PDP^T$, where P is orthogonal and D is diagonal. Now find a diagonal matrix C such that $D = C^T C$, and let $B = PCP^T$. Show that B works.

Question 7. [p 481. #4]

Find the singular values of the matrix $A = \begin{pmatrix} \sqrt{3} & 2 \\ 0 & \sqrt{3} \end{pmatrix}$.

Question 8. [p 481. #13]

Find the Singular Value Decomposition of $A = \begin{pmatrix} 3 & 2 & 2 \\ 2 & 3 & -2 \end{pmatrix}$.

Hint: Work with A^T.

Question 9. [p 492 #7]

Show that an $n \times n$ matrix A is positive definite if and only if A admits a *Cholesky factorization*, that is, $A = R^T R$ for some invertible upper triangular matrix R whose diagonal elements are all positive.

Hint: Use a QR factorization and Question 6 above.

Question 10. [p ???. #??]

Construct the unique positive definite square root of the matrix

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

as follows:

(a) Find the eigenvalues of A, $\lambda_1 \leq \lambda_2 \leq \lambda_3$.

(b) Find an orthonormal basis $\{\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_3\}$ for $\mathbb{R}^3$ consisting of eigenvectors of A.

(c) Find an orthogonal matrix P that diagonalizes A.

(d) Compute $S = PDP^T$, where D is the diagonal matrix whose entries are the nonnegative square roots of the eigenvalues of A.

(e) Compute the matrix

$$B = \sum_{k=1}^{3} \sqrt{\lambda_k}\, \mathbf{p}_k \mathbf{p}_k^T$$

and compare it with S.