MATH 225 Summer 2005
Linear Algebra II
Assignment 4
Due: Wednesday August 3, 2005

Department of Mathematical and Statistical Sciences
University of Alberta

Question 1. [p 382. #24]

Verify the *parallelogram law* for vectors $\mathbf{u}$ and $\mathbf{v}$ in $\mathbb{R}^n$:

$$\|\mathbf{u}+\mathbf{v}\|^2 + \|\mathbf{u}-\mathbf{v}\|^2 = 2\left(\|\mathbf{u}\|^2 + \|\mathbf{v}\|^2\right).$$

Question 2. [p 382. #28]

Suppose that $\mathbf{y}$ is orthogonal to $\mathbf{u}$ and $\mathbf{v}$. Show that $\mathbf{y}$ is orthogonal to every $\mathbf{w}$ in $\text{span}\{\mathbf{u}, \mathbf{v}\}$.

Hint: An arbitrary $\mathbf{w}$ in $\text{span}\{\mathbf{u}, \mathbf{v}\}$ has the form

$$\mathbf{w} = \lambda_1\mathbf{u} + \lambda_2\mathbf{v}.$$

Show that $\mathbf{y}$ is orthogonal to such a vector $\mathbf{w}$.

Question 3. [p 392. #12]

Compute the orthogonal projection of $(1, -1)$ onto the line through $(-1, 3)$ and the origin.

Question 4. [p 393. #25]

Let U be an $m \times n$ matrix with orthonormal columns, and let $\mathbf{x}$ and $\mathbf{y}$ be in $\mathbb{R}^n$, show that

(i) $\|U\mathbf{x}\| = \|\mathbf{x}\|$.

(ii) $(U\mathbf{x}) \cdot (U\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$.

(iii) $(U\mathbf{x}) \cdot (U\mathbf{y}) = 0$ if and only if $\mathbf{x} \cdot \mathbf{y} = 0$.

Question 5. [p 400. #12]

Let $\mathbf{v}_1 = (1, -2, -1, 2)$, $\mathbf{v}_2 = (-4, 1, 0, 3)$, and $\mathbf{y} = (3, -1, 1, 13)$. Find the closest point to $\mathbf{y}$ in the subspace W spanned by $\mathbf{v}_1$ and v_2.

Question 6. [p 401. #24]

Let W be a subspace of $\mathbb{R}^n$ with an orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_p\}$, and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_q\}$ be an orthogonal basis for $W^\perp$.

(i) Explain why $\{\mathbf{w}_1, \ldots, \mathbf{w}_p, \mathbf{v}_1, \ldots, \mathbf{v}_q\}$ is an orthogonal set.

(ii) Explain why the set in part (i) spans $\mathbb{R}^n$.

(iii) Show that $\dim(W) + \dim\left(W^\perp\right) = n$.

Question 7. [p 408. #12]

Given the matrix $A = \begin{pmatrix} 1 & 3 & 5 \\ -1 & -3 & 1 \\ 0 & 2 & 3 \\ 1 & 5 & 2 \\ 1 & 5 & 8 \end{pmatrix}$, find an orthogonal basis for the column space of A.

Question 8. [p 408. #16]

Find a QR factorization of the matrix in Question 7.

Question 9. [p 416. #4]

Let $A = \begin{pmatrix} 1 & 3 \\ 1 & -1 \\ 1 & 1 \end{pmatrix}$, and $\mathbf{b} = \begin{pmatrix} 5 \\ 1 \\ 0 \end{pmatrix}$.

(a) Find a least-squares solution of $A\mathbf{x} = \mathbf{b}$ by constructing the normal equations for $\widehat{\mathbf{x}}$, and

(b) solving for $\widehat{\mathbf{x}}$.

Question 10. [p 417. #24]

Find a formula for the least-squares solution of $A\mathbf{x} = \mathbf{b}$ when the columns of A are orthonormal.