MATH 225 Summer 2005
Linear Algebra II
Assignment 3
Due: Wednesday July 27, 2005

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Question 1. [p 326. #14]

Let $A = \begin{pmatrix} 4 & 0 & -2 \\ 2 & 5 & 4 \\ 0 & 0 & 5 \end{pmatrix}$,

(a) Find the eigenvalues and corresponding eigenvectors of A.

(b) If possible, diagonalize the matrix A, that is find an invertible matrix P and a diagonal matrix D such that $A = PDP^{-1}$.

Question 2. [p 326. #27]

Show that if A is both diagonalizable and invertible, then so is A^{-1}.

Question 3. [p 326. #28]

Show that if an $n \times n$ matrix A has n linearly independent eigenvectors, then so does A^T. [*Hint*: Use the Diagonalization Theorem.]

Question 4. [p 334. #22]

Show that if A is an $n \times n$ matrix which is diagonalizable and B is similar to A, then B is also diagonalizable.

Question 5. [p 334. #24]

Show that if A and B are square matrices which are similar, then they have the same rank.

Question 6. [p 334. #25]

The **trace** of a square matrix A is the sum of the diagonal entries in A and is denoted by $\operatorname{tr}(A)$.

(a) Show that if A and B are any $n \times n$ matrices, then $\operatorname{tr}(AB) = \operatorname{tr}(BA)$.

(b) Show that if A and B are similar, then $\operatorname{tr}(A) = \operatorname{tr}(B)$.

(c) If A is an $n \times n$ matrix with characteristic polynomial

$$p(\lambda) = (-1)^n \left[\lambda^n + a_1\lambda^{n-1} + \cdots + a_{n-1}\lambda + a_n\right],$$

what is the coefficient of λ^{n-1}?

(d) Show that the trace of an $n \times n$ matrix A equals the sum of the eigenvalues of A.

Question 7. [p 341. #23]

Let A be an $n \times n$ real matrix with the property that $A^T = A$, let $\mathbf{x}$ be any vector in $\mathbb{C}^n$ and let $q = \overline{\mathbf{x}}^T A\mathbf{x}$, show that $\overline{q} = q$, that is, q is a real number.

Hint: Justify each of the equalities below:

$$\overline{q} = \overline{\overline{\mathbf{x}}^T A\mathbf{x}} = \mathbf{x}^T \overline{A\mathbf{x}} = \mathbf{x}^T A\overline{\mathbf{x}} = \left(\mathbf{x}^T A\overline{\mathbf{x}}\right)^T = \overline{\mathbf{x}}^T A^T \mathbf{x} = q.$$

Question 8. [p 341. #24]

Let A be an $n \times n$ real symmetric matrix, that is, A has real entries and $A^T = A$. Show that if $A\mathbf{x} = \lambda\mathbf{x}$ for some nonzero vector in $\mathbb{C}^n$, then, in fact, λ is real and the real part of $\mathbf{x}$ is an eigenvector of A.

Hint: Compute $\overline{\mathbf{x}}^T A\mathbf{x}$ and use question 7. Also, examine the real and imaginary parts of $A\mathbf{x}$.

Question 9. [p 371. #2]

Let A and B be $n \times n$ matrices. Show that if $\mathbf{x}$ is an eigenvector of the matrix product AB and $B\mathbf{x} \neq \mathbf{0}$, then $B\mathbf{x}$ is an eigenvector of BA.

Question 10. [p 372. #17]

Let A be a 2×2 real matrix, show that the eigenvalues of A are both real if and only if

$$\left(\operatorname{tr}(A)\right)^2 - 4 \cdot \det(A) \geq 0.$$

Hint: The characteristic polynomial of a 2×2 matrix A is given by

$$p(\lambda) = \lambda^2 - \operatorname{tr}(A) \cdot \lambda + \det(A).$$