MATH 225 Summer 2005
Linear Algebra II
Assignment 2
Due: Wednesday July 20, 2005

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Question 1. [p 270. #28]

Justify the following equalities concerning the four fundamental subspaces of an $m \times n$ matrix A:

$$\dim \text{Row}(A) + \dim \text{Null}(A) = n$$
$$\dim \text{Col}(A) + \dim \text{Null}(A^T) = m$$

Question 2. [p 270. #29]

Show that if A is an $m \times n$ matrix, then the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b} \in \mathbb{R}^m$ if and only if the equation $A^T\mathbf{x} = \mathbf{0}$ has only the trivial solution.

Question 3. [p 308. #16]

Let $A = \begin{pmatrix} 3 & 0 & 2 & 0 \\ 1 & 3 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}$, show that $\lambda = 4$ is an eigenvalue of A and find a basis for the eigenspace corresponding to this eigenvalue.

Question 4. [p 309. #25]

Let λ be an eigenvalue of an invertible matrix A, show that λ^{-1} is an eigenvalue of A^{-1}. [*Hint*: Suppose a nonzero $\mathbf{x}$ satisfies $A\mathbf{x} = \lambda\mathbf{x}$.]

Question 5. [p 309. #30]

Consider an $n \times n$ matrix A with the property that the column sums all equal the same number s. Show that s is an eigenvalue of A. [*Hint*: Show that λ is an eigenvalue of A if and only if it is an eigenvalue of A^T, now do the same problem when all the row sums are equal to s.]

Question 6. [p 309. #34]

Let $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

(a) Find the eigenvalues and corresponding eigenvectors of A.

(b) Solve the difference equation

$$\mathbf{u}_{k+1} = A\mathbf{u}_k,\ k \geq 0$$
$$\mathbf{u}_0 = (1, 0)$$

(c) Use the above to prove Binet's formula for the **Fibonacci sequence**

$$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

for all $n \geq 0$.

Question 7. [p 317. #16]

Let $A = \begin{pmatrix} 5 & 0 & 0 & 0 \\ 8 & -4 & 0 & 0 \\ 0 & 7 & 1 & 0 \\ 1 & -5 & 2 & 1 \end{pmatrix}$.

(a) Find the characteristic polynomial of the matrix A.
(b) Find the eigenvalues and corresponding eigenspaces of A.

Question 8. [p 317. #20]

Use a property of determinants to show that an $n \times n$ matrix A, and its transpose A^T have the same characteristic polynomial and hence the same eigenvalues.

Question 9. [p 317. #24]

Show that if A and B are $n \times n$ matrices that are similar, then $\det A = \det B$.

Question 10. [p ???. #??]

Let A and B be arbitrary 2×2 matrices and let C be their **commutator**, that is, $C = AB - BA$.

(a) Calculate C^2.
(b) Make a conjecture (guess) as to the nature of C^2 for any 2×2 matrices A and B.
(c) Prove your conjecture.