MATH 225 Summer 2005
Linear Algebra II
Assignment 1
Due: Wednesday July 13, 2005

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Question 1. [p 224. #20]

The set of all continuous real-valued functions defined on a closed interval $[a, b]$ in $\mathbb{R}$ is denoted by $C[a, b]$.

(a) Show that $C[a, b]$ is a subspace of the vector space of all real-valued functions defined on $[a, b]$.

(b) Let $V = \{\mathbf{f} \in C[a, b] : \mathbf{f}(a) = \mathbf{f}(b)\}$. Show that V is a subspace of $C[a, b]$.

Question 2. [p 224. #22]

Let F be a fixed 3×2 matrix, and let H be the set of all matrices A in $\mathbb{M}_{2\times4}$ with the property that $FA = 0$ (the zero matrix in $\mathbb{M}_{3\times4}$). Determine if H is a subspace of $\mathbb{M}_{2\times4}$.

Question 3. [p 225. #32]

Let H and K be subspaces of a vector space V. The **intersection** of H and K, written as $H \cap K$, is the set of $\mathbf{v}$ in V that belong to both H and K, that is, $H \cap K = \{\mathbf{v} \in V : \mathbf{v} \in H \text{ and } \mathbf{v} \in K\}$. Show that $H \cap K$ is a subspace of V. Give an example in $\mathbb{R}^2$ to show that the union of two subspaces is not, in general, a subspace.

Question 4. [p 235. #30]

Let $T : V \longrightarrow W$ be a linear transformation from a vector space V into a vector space W. Prove that the range of T is a subspace of W. [*Hint*: Typical elements of the range have the form $T(\mathbf{x})$ and $T(\mathbf{w})$ for some $\mathbf{x}, \mathbf{w}$ in V.]

Question 5. [p 236. #34]

Define $T : C[0, 1] \longrightarrow C[0, 1]$ as follows:
For $\mathbf{f}$ in $C[0, 1]$, let $T(\mathbf{f})$ be the antiderivative $\mathbf{F}$ of $\mathbf{f}$ such that $\mathbf{F}(0) = 0$. Show that T is a linear transformation and describe the kernel of T.

Question 6. [p 236. #36]

Let V and W be vector spaces and let $T : V \longrightarrow W$ be a linear transformation. Let Z be a subspace of W, and let U be the set of all $\mathbf{x}$ in V such that $T(\mathbf{x})$ is in Z. Show that U is a subspace of V. This subspace is usually denoted by $T^{-1}(Z)$.

Question 7. [p 244. #24]

Let $\mathcal{B} = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\}$ be a linearly independent set in $\mathbb{R}^n$. Explain why $\mathcal{B}$ must be a basis for $\mathbb{R}^n$.

Question 8. [p 244. #29]

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k < n$. Explain why S cannot be a basis for $\mathbb{R}^n$.

Question 9. [p 244. #30]

Let $S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ be a set of k vectors in $\mathbb{R}^n$, with $k > n$. Explain why S cannot be a basis for $\mathbb{R}^n$.

Question 10. [p 254. #18]

Let $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ be a basis for a vector space V. Explain why the $\mathcal{B}$-coordinate vectors of $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are the columns $\mathbf{e}_1, \ldots, \mathbf{e}_n$ of the $n \times n$ identity matrix.

Question 11. [p 254. #24]

Let V be a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and the coordinate mapping $\mathbf{x} \longrightarrow [\mathbf{x}]_{\mathcal{B}}$. Show that the coordinate map is *onto* $\mathbb{R}^n$, that is, given any $\mathbf{y}$ in $\mathbb{R}^n$, with entries $y_1, \ldots, y_n$, produce a $\mathbf{u}$ in V such that $[\mathbf{u}]_{\mathcal{B}} = \mathbf{y}$.

Question 12. [p 254. #25]

Let V be a vector space with basis $\mathcal{B} = \{\mathbf{b}_1, \ldots, \mathbf{b}_n\}$ and the coordinate mapping $\mathbf{x} \longrightarrow [\mathbf{x}]_{\mathcal{B}}$. Show that a subset $\{\mathbf{u}_1, \ldots, \mathbf{u}_p\}$ in V is linearly independent if and only if the set of coordinate vectors $\{[\mathbf{u}_1]_{\mathcal{B}}, \ldots, [\mathbf{u}_p]_{\mathcal{B}}\}$ is linearly independent in $\mathbb{R}^n$.
Hint: Explain why, since the coordinate mapping is one-to-one, the following equations have the same solutions, $c_1, \ldots, c_p$.

$$\begin{aligned} c_1\mathbf{u}_1 + \cdots + c_p\mathbf{u}_p &= \mathbf{0} \qquad \text{(zero vector in } V) \\ c_1[\mathbf{u}_1]_{\mathcal{B}} + \cdots + c_p[\mathbf{u}_p]_{\mathcal{B}} &= [\mathbf{0}]_{\mathcal{B}} \quad \text{(zero vector in } \mathbb{R}^n) \end{aligned}$$

Question 13. [p 261 #21]

The first four **Hermite polynomials** are 1, $2t$, $-2+4t^2$, and $-12t+8t^3$. These polynomials arise naturally in the study of certain important differential equations in mathematical physics. Show that the first four Hermite polynomials form a basis for $\mathbb{P}_3$.

Question 14. [p 261. #22]

The first four **Laguerre polynomials** are 1, $1-t$, $2-4t+t^2$, and $6-18t+9t^2-t^3$. Show that these polynomials form a basis for $\mathbb{P}_3$.

Question 15. [p 262. #28]

Show that the space $C(\mathbb{R})$ of all continuous functions defined on the real line is an infinite-dimensional vector space.