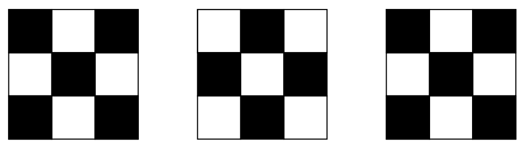
math22

Solutions to Assignment 5

Problem 1. A mouse eats its way through a $3 \times 3 \times 3$ cube of cheese by eating all of the $1 \times 1 \times 1$ subcubes. If it starts at a corner subcube and always moves on to an adjacent subcube (sharing a face of area 1), can it do this and eat the center subcube last? Give a method for doing this or prove it is impossible. (Ignore gravity.)

SOLUTION: Color the subcubes black (B) and white (W) according to the scheme shown below.



Top Middle Bottom

Then any path joining a corner subcube with the center subcube which always moves to an adjacent subcube must have the form

$$BWBWBW \cdots BW$$

that is, the number of B's must equal the number of W's. Therefore, since there are 27 subcubes, and 27 is an odd integer, there cannot exist a path which goes from a corner subcube and ends up at the center subcube passing through every subcube exactly once.

Problem 2. Give a combinatorial argument to find a closed form expression for the following sequence

$$a_n = 2 \cdot 1 \cdot \binom{n}{2} + 3 \cdot 2 \cdot \binom{n}{3} + \cdots + n \cdot (n-1) \cdot \binom{n}{n}$$

for $n = 2, 3, 4, \ldots$.

Hint: Given n people, count the number of ways to form an Ed Leonard fan club which contains a president and a vice-president.

SOLUTION: If we choose the president and vice-president first, we have n choices for the president, and for each of these n choices we have $n-1$ choices for the vice-president. Thus, there are $n(n-1)$ ways to choose the president and the vice-president.

For each of the remaining $n-2$ fans, we have 2 choices, either they are in the club or they are not in the club, so we have 2^{n-2} ways to choose the remaining members of the fan club.

Therefore, the number of ways to form an Ed Leonard fan club which contains a president and a vice-president is

$$n(n-1) \cdot 2^{n-2}.$$

On the other hand, if the fan club has k members, where $2 \le k \le n$, we can choose the k members of the fan club first in $\binom{n}{k}$ ways, and then from among these k we choose the president and vice-president in $k(k-1)$ ways.

Thus, the number of ways of forming an Ed Leonard fan club is also equal to

$$\sum_{k=2}^{n} k(k-1)\binom{n}{k} = 2\cdot 1\cdot \binom{n}{2} + 3\cdot 2\cdot \binom{n}{3} + \cdots + n\cdot(n-1)\cdot\binom{n}{n}.$$

Since these have to be the same, the closed form expression for a_n is given by

$$a_n = n(n-1)\cdot 2^{n-2}$$

for $n = 2, 3, 4, \ldots$.

Problem 3. A town jail contains four holding cells. On a particularly busy night, twelve people are arrested. Certain prisoners do not get along with certain others and must be put into separate cells, as shown in the following table.

Prisoner	doesn't get along with
1	3, 5, 8, 9, 10, 11
2	3, 4, 6, 7, 9, 11
3	1, 2, 6, 8, 11, 12
4	2, 5, 6, 8, 10,12
5	1, 4, 7, 9, 10
6	2, 3, 4, 7, 9, 11, 12
7	2, 5, 6, 8, 10
8	1, 3, 4, 7, 12
9	1, 2, 5, 6, 11
10	1, 4, 5, 7, 12
11	1, 2, 3, 6, 9
12	3, 4, 6, 8, 10

(a) Find a way of putting the prisoners into the four cells in such a way as to avoid possible conflicts during the night.

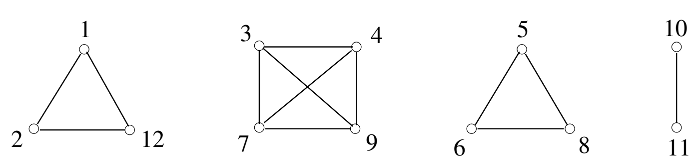
(b) Draw the graph G whose vertices correspond to the prisoners, with an edge between two vertices if and only if the two prisoners share a cell for the night as in part (a).

SOLUTION:

(a) One way to keep the peace is shown in the table below.

Cell #	Contains Prisoners
1	1, 2, 12
2	3, 4, 7, 9
3	5, 6, 8
4	10, 11

(b) The graph G corresponding to this distribution of prisoners into holding cells is shown below.



Problem 4. Let G be a graph with p vertices, show that G has $2^p - 1$ induced subgraphs.

SOLUTION: The edges of an induced subgraph H of G are completely determined once we specify the vertices in H, and since the vertex set $V(H)$ must contain at least one vertex and at most p vertices, then the number of induced subgraphs is

$$\sum_{k=1}^{p} \binom{p}{k} = 2^p - 1.$$

Problem 5. Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let $p = |V|$ be the number of vertices in G, and $q = |E|$ the number of edges in G. The **average degree** of the vertices in G is defined to be

$$A(G) = \frac{1}{p} \sum_{v \in V} \deg(v).$$

If G is a connected graph, what can you say about G if

(a) $A(G) > 2$? (b) $A(G) = 2$? (c) $A(G) < 2$?

Draw a few pictures before committing yourself!!!

SOLUTION: First note that

$$A(G) = \frac{1}{p} \sum_{v \in V} \deg(v) = \frac{2q}{p}$$

(a) $A(G) > 2$ if and only if $q > p$.

Suppose that G is a connected graph and $q > p$, then G is not a tree and hence contains a cycle. Remove an edge from the cycle, then the resulting graph H is still connected, has p vertices and $q - 1$ edges.

Now, $p < q = (q - 1) + 1$, so that H is not a tree, so it also has a cycle. Therefore, G has at least two cycles.

Conversely, suppose that G is a connected graph which has at least two cycles, then we may remove an edge from two of the cycles and the resulting graph H is still connected, so that

$$p \leq (q - 2) + 1 = q - 1 < q,$$

and $A(G) > 2$.

Thus, if G is a connected graph, then $A(G) > 2$ if and only if G has at least two cycles.

(b) $A(G) = 2$ if and only if $q = p$.

Suppose that G is a connected graph and $q = p$, then G is not a tree and hence contains a cycle. Remove an edge from the cycle, then the resulting graph H is still connected, has p vertices and $q - 1$ edges and $p = (q - 1) + 1$, so that H is a tree. Therefore, G has exactly one cycle.

Conversely, suppose that G is a connected graph with exactly one cycle, if we remove an edge from the cycle, the resulting graph H is still connected and has no cycles, hence is a tree. Therefore, since H has p vertices and $q - 1$ edges, we have $p = (q - 1) + 1 = q$.

Thus, if G is a connected graph, then $A(G) = 2$ if and only if G has exactly one cycle.

(c) $A(G) < 2$ if and only if $q < p$.

Suppose that G is a connected graph and $q < p$, then since p and q are integers, we must have $q + 1 \leq p$. We showed in class that any connected graph with p vertices and q edges has $p \leq q + 1$, therefore $p = q + 1$ and G is a tree.

Conversely, suppose that G is a tree, then G is connected and $p = q + 1$, so that

$$A(G) = \frac{2q}{p} = \frac{1}{p}(2p - 2) = 2 - \frac{2}{p} < 2.$$

Thus, if G is a connected graph, then $A(G) < 2$ if and only if G is a tree.

Problem 6. Given a graph G, show that the following are equivalent

(a) G is a tree.

(b) G is connected, and the removal of any edge disconnects G.

(c) G has no cycles, and the addition of any new edge creates exactly one cycle.

SOLUTION:

$(a) \implies (b)$ Suppose that G is a tree, then G is connected. Suppose that we remove an edge $e = ab$ from G and the resulting graph $G - e$ is also connected, then there is a path from a to b in $G - e$, and if we replace the edge $e = ab$, then G contains a cycle. This is a contradiction, since we assumed that G is a tree. Therefore G is connected and the removal of any edge disconnects G.

$(b) \implies (a)$ Suppose that G is connected and the removal of any edge disconnects G. If G contains a cycle, then we can remove an edge from this cycle, and the remaining graph is also connected, which is a contradiction. Therefore G has no cycles, hence G is a tree.

$(a) \implies (c)$ Suppose that G is a tree, then G is connected. If we add an edge e to the graph G, the resulting graph $G + e$ is not a tree, since we can remove the edge e from the connected graph $G + e$ without disconnecting the graph. Therefore the graph $G + e$ must contain a cycle. If the addition of the edge $e = ab$ created more than one cycle, then in the original graph G, there must have been two distinct paths joining a and b, which is a contradiction, since G is a tree. Therefore, if G is a tree then G has no cycles and the addition of any new edge creates exactly one cycle.

$(c) \implies (a)$ Let G be a graph which has no cycles and such that the addition of any new edge creates exactly one cycle. Suppose that G is not connected, then it has at least two connected components all of which are trees. If we add an edge e joining a vertex a in one component G_1 to a vertex b in another component G_2, then by hypothesis we create exactly one cycle. Since there were no other edges joining vertices in G_1 to vertices in G_2, this edge ab must be in the cycle. However, this implies that there is a path from a to b which does not include the edge ab, that is, a and b belong to the same connected component, which is a contradiction. Therefore, G is connected and so G is a tree.

Problem 7. Let $G = (V, E)$ be a connected graph. The graph G is said to be **bipartite** if and only if there is a partition of the vertex set $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, such that for every edge $ab \in E$, the end vertices are in different sets, that is, either $a \in V_1$ and $b \in V_2$, or $a \in V_2$ and $b \in V_1$.

(a) Show that G is two colorable if and only if it is bipartite.

(b) Show that G is two colorable if and only if it contains no cycles of odd length.

SOLUTION:

(a) Let $G = (V, E)$ be a simple undirected and connected graph.

  (i) If G is two colorable, then color the vertices B and W so that no two adjacent vertices have the same color. Let $V_1 =$ set of all B vertices and $V_2 =$ set of all W vertices, then $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$, and all edges in $E(G)$ go from a vertex in V_1 to a vertex in V_2, thus, G is bipartite.

  (ii) If G is bipartite, with bipartition sets V_1 and V_2, color the vertices in V_1 blue and color the vertices in V_2 white, then this is a proper two coloring for G, and G is two colorable.

(b) Let $G = (V, E)$ be a connected graph, for any two vertices u, $v \in V$, we let $d(u, v)$ be the length of a shortest path joining u and v.

(i) If G is two colorable, and G contains a cycle C, then C is also two colorable, and so C must have even length, thus, G contains no cycles of odd length.

(ii) Conversely, suppose that G has no cycles of odd length, let $v_0 \in V(G)$ be a fixed vertex in G, and define

$$\begin{aligned} \text{Color}(v) = 0 \qquad &\text{if} \quad d(v_0, v) \quad \text{is even} \\ \text{Color}(v) = 1 \qquad &\text{if} \quad d(v_0, v) \quad \text{is odd.} \end{aligned}$$

We will show that this is a valid two coloring of G.

Let x and y be adjacent vertices in G, and choose a shortest path from v_0 to x and choose a shortest path from v_0 to y. Let u be the last common vertex in these two shortest paths, here u may be equal to v_0, x, or y. Now consider $d(u, x)$ and $d(u, y)$:

If $u = x$ or $u = y$, then either $d(u, x) = d(u, y) + 1$ or $d(u, y) = d(u, x) + 1$. In either case, one is even and the other is odd, that is, they have different parity,

If u is not one of x or y, we can compute the length of the cycle containing the vertices u, x, and y as $d(u, x) + 1 + d(u, y)$, and we know this is even, so again $d(u, x)$ and $d(u, y)$ have different parity.

Now, since

$$d(v_0, x) = d(v_0, u) + d(u, x)$$

and

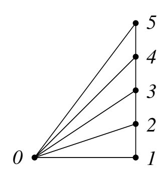
$$d(v_0, y) = d(v_0, u) + d(u, y),$$

then $d(v_0, x)$ and $d(v_0, y)$ have different parity. Therefore, x and y receive different colors, and G is two colorable.

Note: Since 0 is an even integer and a tree has no cycles, the above shows that any tree is bipartite. Try to give a direct proof of this fact.

Problem 8. A *fan* of order n is a graph on $n + 1$ vertices, labeled $\{0, 1, 2 \ldots, n\}$, with $2n - 1$ edges defined as follows: Vertex 0 is connected by an edge to each of the other n vertices, and vertex k is connected by an edge to vertex $k + 1$, for $1 \le k < n$.

For example, the fan of order 5, which has six vertices and nine edges, is shown below.



Let a_n be the number of spanning trees for a fan of order n.

(a) Calculate a_1, a_2, and a_3, and show all the spanning trees in each case.

(b) By observing how the topmost vertex (vertex n) is connected to the rest of the spanning tree, show that a_n satisfies the full-history recurrence relation

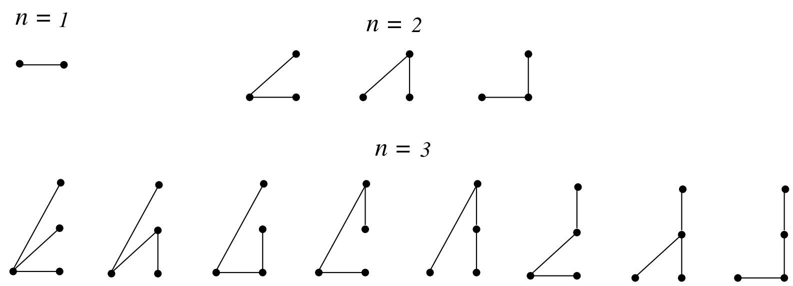
$$a_n = 1 + a_{n-1} + \sum_{k=1}^{n-1} a_k$$

for $n \ge 1$, where $a_0 = 0$ and $a_1 = 1$.

(c) Conjecture a value for a_n, for $n \ge 1$, and prove your conjecture is true.

Solution: Let a_n be the number of spanning trees for a fan of order n.

(a) The spanning trees for $n = 1$, $n = 2$, $n = 3$ are shown below

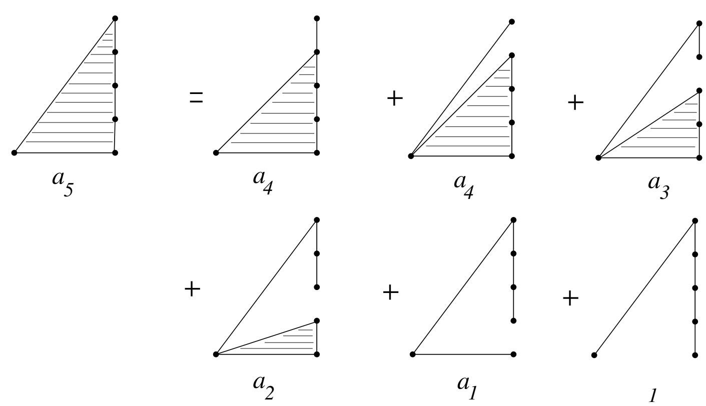


and is clear that $a_1 = 1$, $a_2 = 3$, and $a_3 = 8$.

(b) Given a fan of order n, if the topmost vertex is not connected to vertex 0, then it must be connected to vertex $n-1$, since the graph is connected. In this case, any of the a_{n-1} spanning trees for the remaining fan (on the vertices 0 through $n-1$) will complete a spanning tree for the entire graph.

If the topmost vertex is connected to vertex 0, then there exists an integer k, with $k \leq n$, such that the vertices n, $n-1$, …, k are connected directly to vertex 0, but the edge between k and $k-1$ is not in the subtree. In this case, there cannot be any edges between vertex 0 and the vertices $\{n-1, \ldots, k\}$, or there would be a cycle.

If $k = 1$, the subtree is uniquely determined, and if $k > 1$, then any of the a_{k-1} ways to produce a spanning tree on $\{0, 1, \ldots, k-1\}$ will give a spanning tree for the entire graph. For example, when $n = 5$, we have



Therefore, in general,

$$a_n = a_{n-1} + a_{n-1} + a_{n-2} + a_{n-3} + \cdots + a_1 + 1,$$

that is,

$$a_n = a_{n-1} + \sum_{k=1}^{n-1} a_k + 1,$$

and if we define $a_0 = 0$, then

$$a_n = a_{n-1} + \sum_{k=0}^{n-1} a_k + 1 \qquad (*)$$

for all $n \geq 1$.

(c) To solve this problem, we can find another recurrence relation satisfied by the sequence $\{a_n\}_{\geq 0}$ as follows, we write down the recurrence relation for n and also for $n+1$, and subtract the first from the second to get

$$a_{n+1} - a_n = a_n - a_{n-1} + a_n,$$

that is,

$$a_{n+1} = 3a_n - a_{n-1}$$

together with the initial conditions $a_0 = 0$ and $a_1 = 1$, we get the discrete initial value problem

$$\begin{aligned} a_{n+1} &= 3a_n - a_{n-1}, \quad n \geq 1 \\ a_0 &= 0, \\ a_1 &= 1. \end{aligned}$$

Now note that for the sequence $\{F_{2n}\}_{n\geq 0}$, we have

$$F_{2n+2} = F_{2n+1} + F_{2n} = 2F_{2n} + F_{2n-1} = 2F_{2n} + F_{2n} - F_{2n-2},$$

and letting $b_n = F_{2n}$ for $n \geq 0$, then the sequence $\{b_n\}_{n\geq 0}$ satisfies the same discrete initial value problem,

$$\begin{aligned} b_{n+1} &= 3b_n - b_{n-1}, \quad n \geq 1 \\ b_0 &= 0, \\ b_1 &= 1. \end{aligned}$$

An easy induction argument now shows that $a_n = F_{2n}$ for all $n \geq 0$.