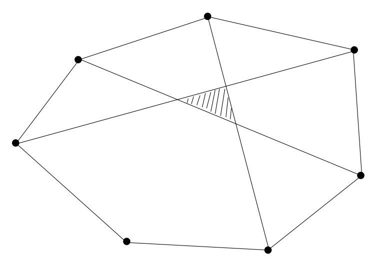
math22

Solutions to Assignment 4

Problem 1. How many triangles are formed using chords and sides of a convex $n-$gon, where the vertices of the triangle need not be vertices of the $n-$gon?

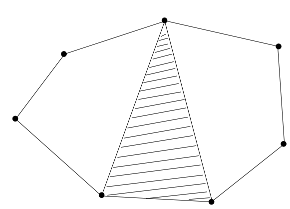
For example, one such triangle is shown below.



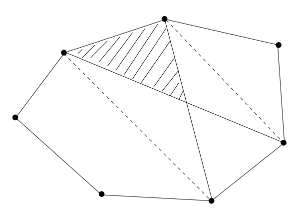
Assume that no three chords meet at the same interior point.

Hint: Relate different sorts of triangles to different sized sets of vertices of the polygon.

SOLUTION: The number of triangles with 3 vertices on the polygon is $\binom{n}{3}$.

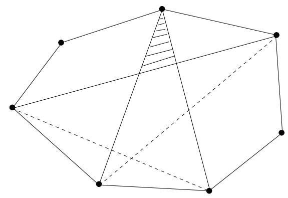


The number of triangles with 2 vertices on the polygon is $4\binom{n}{4}$;



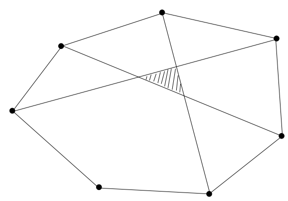
the sides to the third triangle vertex extend to two other polygon vertices, and the chords from each set of 4 polygon vertices create four such triangles.

The number of triangles with 1 vertex on the polygon is $5\binom{n}{5}$;



the sides extend to four other polygon vertices, and the chords from each set of 5 polygon vertices create five such triangles.

The number of triangles with 0 vertices on the polygon is $\binom{n}{6}$;



the sides extend to six polygon vertices, and the chords from each set of 6 polygon vertices create exactly one such triangle.

Thus, the number of triangles formed using the chords and sides of a convex n-gon is

$$\binom{n}{3} + 4\binom{n}{4} + 5\binom{n}{5} + \binom{n}{6}.$$

Problem 2. The *Fibonacci Sequence:*

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \cdots$$

where each term is the sum of the two preceding terms, satisfies the recurrence relation

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n \\ F_0 &= 0 \\ F_1 &= 1 \end{aligned}$$

for $n = 0, 1, 2, \ldots$.

(a) Show that

$$F_{n+m+1} = F_{m+1}F_{n+1} + F_m F_n$$

for all $m, n \geq 0$.

(b) Show that

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$$

for all $n \geq 1$.

(c) Show that

$$F_{2n} = \binom{n}{0}F_0 + \binom{n}{1}F_1 + \binom{n}{2}F_2 + \binom{n}{3}F_3 + \cdots + \binom{n}{n}F_n$$

for all $n \geq 0$.

SOLUTION:

(a) Letting $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$, an easy induction argument shows that

$$A^k = \begin{pmatrix} F_{k+1} & F_k \\ F_k & F_{k-1} \end{pmatrix}$$

for all integers $k \geq 1$.

Therefore,

$$A^{m+m} = \begin{pmatrix} F_{m+n+1} & F_{m+n} \\ F_{m+n} & F_{m+n-1} \end{pmatrix}$$

for all positive integers m and n. On the other hand, $A^{m+n} = A^m \cdot A^n$, so that

$$A^{m+n} = \begin{pmatrix} F_{m+1} & F_m \\ F_m & F_{m-1} \end{pmatrix} \begin{pmatrix} F_{n+1} & F_n \\ F_n & F_{n-1} \end{pmatrix} = \begin{pmatrix} F_{m+1}F_{n+1} + F_m F_n & F_{m+1}F_n + F_m F_{n-1} \\ F_m F_{n+1} + F_{m-1}F_n & F_m F_n + F_{m-1}F_{n-1} \end{pmatrix}$$

and equating the entries in the second row and the first column, we have

$$F_{m+n} = F_m F_{n+1} + F_{m-1} F_n.$$

(b) In part (a), replace m by n and n by $n-1$ to get

$$F_{2n-1} = F_n^2 + F_{n-1}^2. \tag{1}$$

Next, replace m by n in part (a) to get

$$F_{2n} = F_n F_{n+1} + F_{n-1} F_n = F_{n+1}\big[F_{n+1} - F_{n-1}\big] + F_{n-1}\big[F_{n+1} - F_{n-1}\big] = F_{n+1}^2 - F_{n-1}^2. \tag{2}$$

Finally, replace m by $2n$ in part (a) and use (1) and (2) to get

$$\begin{aligned} F_{3n} &= F_{2n}F_{n+1} + F_{2n-1}F_n \\ &= F_{n+1}^3 - F_{n+1}F_{n-1}^2 + F_n^3 + F_n F_{n-1}^2 \\ &= F_{n+1}^3 + F_n^3 - F_{n-1}^2\big[F_{n+1} - F_n\big] \\ &= F_{n+1}^3 + F_n^3 - F_{n-1}^3. \end{aligned}$$

(c) From Binet's formula for the Fibonacci sequence we have

$$F_k = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]$$

for $k \geq 0$. Therefore, from the binomial theorem, for $n \geq 0$ we have

$$\begin{aligned} \sum_{k=0}^{n} \binom{n}{k} F_k &= \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1+\sqrt{5}}{2}\right)^k - \frac{1}{\sqrt{5}} \sum_{k=0}^{n} \binom{n}{k} \left(\frac{1-\sqrt{5}}{2}\right)^k \\ &= \frac{1}{\sqrt{5}}\left(1 + \frac{1+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(1 + \frac{1-\sqrt{5}}{2}\right)^n \\ &= \frac{1}{\sqrt{5}}\left(\frac{3+\sqrt{5}}{2}\right)^n - \frac{1}{\sqrt{5}}\left(\frac{3-\sqrt{5}}{2}\right)^n \end{aligned}$$

However,

$$\frac{3+\sqrt{5}}{2} = \left(\frac{1+\sqrt{5}}{2}\right)^2 \quad \text{and} \quad \frac{3-\sqrt{5}}{2} = \left(\frac{1-\sqrt{5}}{2}\right)^2,$$

so that

$$\sum_{k=0}^{n} \binom{n}{k} F_k = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{2n} = F_{2n}.$$

Problem 3. There are $2n$ people standing in line at a box office. Admission is one dollar and n of the people have exactly this amount. The other n each have exactly one two dollar coin. Unfortunately, the box office starts off with no change. A sequence of these $2n$ people is said to be *workable* if, up to each point, the number of people with one dollar is not less than the number of people with 2 dollars. In such situations correct change can be given to each person who needs it. How many workable situations are there?

Hint: Let L stand for anyone with a dollar and T stand for anyone with two dollars. The total number of permutations of n T's and n L's is

$$\binom{2n}{n}$$

since each arrangement is determined by which of the $2n$ possible locations are chosen for the n T's. Now count the number of nonworkable permutations and subtract this from the total number to get the number of workable permutations.

Solution: Note that the sequence

$$L\ L\ T\ T\ T\ L$$

is not workable, since by the time the third T wants change, there is none left, and this is typical of a nonworkable sequence, there is a first T who cannot receive change. We will use this fact to count the number of nonworkable sequences.

If we have a nonworkable permutation, the first snag occurs at some T that is preceded by an equal number of T's and L's, say m of each. This T occurs at the $2m+1^{\text{st}}$ term.

If we take the first $2m+1$ terms and reverse them, that is, replace each T by L and each L by T, then the whole permutation now has $n+1$ L's and $n-1$ T's (the snag becomes an L and the equal number of T's and L's before it are just reversed). Every nonworkable sequence becomes, by this process, a different sequence of $n+1$ L's and $n-1$ T's

Also, every permutation of $n+1$ L's and $n-1$ T's can be readjusted back into some nonworkable sequence by noting the first time the L's outnumber the T's by 1 (since there are more L's than T's this must happen) and reversing the sequence up to and including that pivotal L.

The number of permutations of $n+1$ L's and $n-1$ T's is given by $\binom{2n}{n-1}$ since each is determined by the choice of the $n-1$ locations for the T's. By the correspondence described above, this is the same as the number of nonworkable sequences.

Therefore, the number of workable permutations is

$$\begin{aligned}\binom{2n}{n} - \binom{2n}{n-1} &= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n+1)!\,(n-1)!} = \frac{(2n)!}{n!\,(n-1)!}\left[\frac{1}{n} - \frac{1}{n+1}\right] \\ &= \frac{(2n)!}{n!\,(n-1)!}\frac{1}{n(n+1)} = \frac{1}{n+1}\frac{(2n)!}{n!\,n!} \\ &= \frac{1}{n+1}\binom{2n}{n}\end{aligned}$$

that is, the number of workable permutations is the n^{th} *Catalan number* $\frac{1}{n+1}\binom{2n}{n}$.

Problem 4. For $n \geq 0$, let a_n be the number of regions on the surface of a sphere formed by n great circles, no three of which are concurrent.

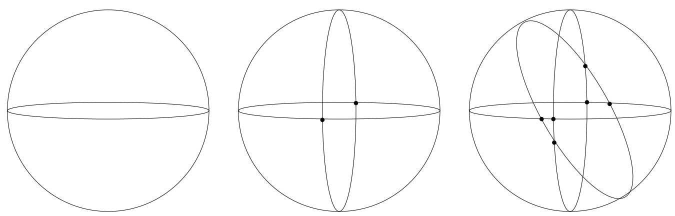
(a) What are $a_0,\ a_1,\ a_2,$ and a_3 ?

(b) Find a recurrence relation and initial condition satisfied by a_n.

(c) Solve the recurrence relation you found in part (b).

SOLUTION:

(a) From the figure below it is obvious that $a_0 = 1,\ a_1 = 2,\ a_2 = 4$ and $a_3 = 8$.



(b) It looks at first like $a_n = 2^n$ for all $n \geq 0$, but this is not the case.

Suppose that n circles have been drawn on the surface of the sphere so that no three are concurrent, and suppose that an $(n+1)^{\text{st}}$ circle is added so that no three of the $n+1$ circles are concurrent.

The new circle meets each of the old circles in two points, making $2n$ points of intersection on the new circle, and these $2n$ points are all different since no three of the circles are concurrent.

The $2n$ points divide the new circle into $2n$ arcs. Each of these arcs divides one of the existing regions into two parts, so there are

$$a_n + 2n$$

regions formed by the $n+1$ great circles. Therefore, a_n satisfies the recurrence relation and initial value

$$\begin{aligned} a_{n+1} &= a_n + 2n, \quad n \geq 1 \\ a_1 &= 2. \end{aligned}$$

(c) For each integer k with $1 \leq k \leq n-1$, we have

$$a_{k+1} = a_k + 2k,$$

so that

$$a_n - a_1 = \sum_{k=1}^{n-1} (a_{k+1} - a_k) = \sum_{k=1}^{n-1} 2k = n(n-1),$$

and therefore

$$a_n = n(n-1) + 2$$

for all $n \geq 1$.

Problem 5. Consider a party attended by n married couples. Suppose that no person shakes hands with his or her spouse, and the $2n-1$ people other than the host shake hands with different numbers of people. With how many people does the hostess shake hands?

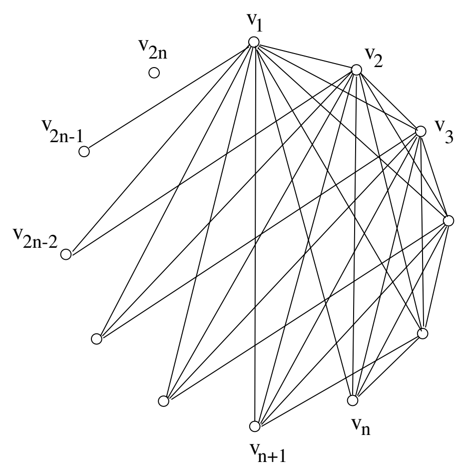
SOLUTION: Let the $2n$ people be denoted by the set $V = \{v_1, v_2, \ldots, v_n, v_{n+1}, \ldots, v_{2n}\}$ and represent the situation described in the problem statement by a graph $G = (V, E)$ where there is an edge joining v_i and v_j if and only if v_i and v_j have shaken hands.

The degree sequence for the graph, $d_1 \geq d_2 \geq \cdots \geq d_{2n}$, is given by

$$2n-2,\ 2n-3,\ \cdots,\ n+1,\ n,\ \cdots,\ 2,\ 1,\ 0$$

Exactly one of these degrees is repeated, since from the pigeon-hole principle, there must be at least two vertices with the same degree, and excluding the host, all of the remaining $2n-1$ vertices have different degrees.

Now, we construct a graph G having the degree sequence above



and note the following:

- Since $\deg(v_1) = 2n-2$, then v_1 has shaken hands with every one except himself and his spouse, thus v_1 and v_{2n} must be married.
- Since $\deg(v_2) = 2n-3$, then v_2 has shaken hands with $v_1, v_3, v_4, \ldots, v_{2n-2}$, and therefore v_2 and v_{2n-1} must be married.
- Since $\deg(v_3) = 2n-4$, then v_3 has shaken hands with $v_1, v_2, v_4, \ldots, v_{2n-3}$, and therefore v_3 and v_{2n-2} must be married.

Continuing in this way, we see that for each k with $1 \leq k \leq n-1$, we must have v_k married to v_{2n-k+1}. Thus, $v_1, v_2, \ldots, v_{n-1}$ are married, respectively, to $v_{2n}, v_{2n-1}, \ldots, v_{n+2}$; but this means that v_n and v_{n+1} must be married, since they are the only two left.

Finally, by construction, v_n is adjacent to v_k for $1 \leq k \leq n-1$, so that $\deg(v_n) = n-1$. Similarly, v_{n+1} is adjacent to v_k for $1 \leq k \leq n-1$, therefore $\deg(v_{n+1}) = n-1$ also.
Since the degree $n-1$ is repeated, then either v_n or v_{n+1} is the host and the other is the hostess. In either case, the hostess has shaken hands $n-1$ times.

Problem 6. Let G be a graph whose vertices correspond to the bit-strings of length n, $a = a_1a_2 \cdots a_n$ where $a_i = 0$ or 1, and whose edges are formed by joining those bit-strings which differ in exactly two places.

(a) Show that G is regular, that is, every vertex has the same degree, and find the degree of each vertex.

(b) Find a necessary and sufficient condition that there exist a path joining two vertices $a = a_1a_2 \cdots a_n$ and $b = b_1b_2 \cdots b_n$ in G.

(c) Find the number of connected components of G.

Solution:

(a) The graph G has 2^n vertices, since this is the number of bit-strings of length n. Also, given any vertex $u = u_1u_2 \cdots u_n$, a vertex $v = v_1v_2 \cdots v_n$ is adjacent to u if and only if the Hamming distance between them is exactly 2, that is, they differ in exactly two bits. Thus the number of vertices adjacent to u is just the number of ways to choose the two bits in which they differ, that is, $\deg(u) = \binom{n}{2} = \dfrac{n(n-1)}{2}$ for each vertex u in the graph G.

(b) Note that in order to move from $a = a_1a_2 \cdots a_n$ to an adjacent vertex $b = b_1b_2 \cdots b_n$ we need to change two of the bits in a. If the bits we change are both 0 or they are both 1, then the number of 1's remains even or odd. Also, if one of the bits we change is a 0 and the other is a 1, then again, the number of 1's remains even or odd. Therefore, as we move from one vertex to an adjacent vertex, the parity doesn't change, that is, the number of $1's$ in the bit-strings remains even or odd. It follow that there is a path joining the vertices a and b if and only if the bit strings have the same parity.

(c) The number of connected components in G is two, since all vertices with an odd number of 1's are in one component, and all vertices with an even number of 1's are in another component.

Problem 7. The queen and her prime minister each live in a complex of underground rooms.
The queen's rooms are 15 in number, 1 for her and 14 for her servants, and they are connected by tunnels. There is at most one tunnel between any two rooms.
For each of the servant's rooms, there is one and only one path that leads to the queen's room.
The prime minister and his cabinet occupy 7 rooms, none of which are the queen's rooms. There is at most one tunnel between any two of the prime minister's rooms.
Together, the underground complexes have a total of 36 tunnels.
Explain why the entire complex is connected.

Solution: Since there is at most one tunnel between any two rooms, we can represent the complex by a graph whose vertices correspond to the rooms and whose edges correspond to the tunnels between rooms.

If the entire complex is not connected, then there are at least two connected components. The queen's complex is connected, since each of the nodes corresponding to the servant's rooms is in the same connected component containing the node corresponding to the queen's room. Also, the queen's complex contains no cycles, since for each of the servant's rooms there is exactly one path that leads to the queen's room. Thus, the queen lives in a tree which has 15 vertices and $15 - 1 = 14$ edges.
Since there are a total of 26 edges in the graph, the prime minister's complex would have 7 vertices and $26 - 14 = 22$ edges. Thus, if there were no edge from the queen's complex to the prime minister's complex, the connected component containing the prime minister's rooms would have 7 vertices and 22 edges. This is a contradiction since K_7 has the maximum number of edges for a graph with 7 vertices, namely,

$$\binom{7}{2} = \frac{7 \cdot 6}{2} = 21$$

edges. Therefore the entire complex is connected.

Problem 8. By using various combinations of the red, green, and blue filters on a spotlight the lighting technician at a theatre can obtain 8 lighting effects. The filters may be changed one at a time by either adding one or removing one. Starting and ending with no filters, how can the technician test all the effects without repeating any effect except the final one.

SOLUTION: The technician can label the combination of filters as (r, g, b), where each of r, g, or b can be 0 or 1, and then use the graph below.