math 222

Solutions to Assignment 3

Problem 1.

(a) Find a closed form expression for

$$1 + 2 + 3 + \cdots + n.$$

(b) Make a conjecture about the terms of the following sequence, and prove your conjecture.

$$\frac{1}{1+2}, \qquad \frac{1+2}{2+3+4}, \qquad \frac{1+2+3}{3+4+5+6}, \qquad \frac{1+2+3+4}{4+5+6+7+8}, \qquad \ldots$$

Solution:

(a) We have

$$\begin{aligned} S_n &= 1 + 2 + \cdots + n - 1 + n \\ S_n &= n + n - 1 + \cdots + 2 + 1 \end{aligned}$$

and adding, we get

$$2S_n = (1+n)+(2+n-1)+\cdots+(n-1+2)+(n+1) = (n+1)+(n+1)+\cdots+(n+1)+(n+1) = n(n+1),$$

so that $S_n = \dfrac{n(n+1)}{2}$.

(b) Since each of the terms shown above is equal to $\frac{1}{3}$, if we let

$$a_n = \frac{1 + 2 + \cdots + n}{n + n + 1 + \cdots + 2n},$$

for $n \geq 1$, then it appears that $a_n = \frac{1}{3}$ for all $n \geq 1$.
In order to see that this is indeed the case, we use the result from part (a) to write

$$a_n = \frac{\frac{1}{2}n(n+1)}{\frac{1}{2}2n(2n+1) - \frac{1}{2}n(n-1)} = \frac{n+1}{4n + 2 - (n-1)} = \frac{n+1}{3n+3} = \frac{n+1}{3(n+1)} = \frac{1}{3}$$

for $n \geq 1$.

Problem 2. Find a closed form expression for

$$a_n = 1^5 + 2^5 + \cdots + n^5$$

for $n \geq 1$.

SOLUTION: From the binomial theorem we have

$$(k+1)^5 - k^5 = 5k^4 + 10k^3 + 10k^2 + 5k + 1$$

for each $k = 1, 2, \ldots, n$. Adding these equations, the sum on the left-hand side telescopes and we get

$$(n+1)^5 - 1^5 = 5\sum_{k=1}^{n} k^4 + 10\sum_{k=1}^{n} k^3 + 10\sum_{k=1}^{n} k^2 + 5\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1,$$

and since

$$\sum_{k=1}^{n} 1 = n, \quad \sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \quad \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}, \quad \text{and} \quad \sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4},$$

then we have

$$(n+1)^5 = 5\sum_{k=1}^{n} k^4 + \frac{10}{4}n^2(n+1)^2 + \frac{10}{6}n(n+1)(2n+1) + \frac{5}{2}n(n+1) + n + 1,$$

so that

$$\begin{aligned} 5\sum_{k=1}^{n} k^4 &= \frac{(n+1)}{6}\left[6(n+1)^4 - 15n^2(n+1) - 10n(2n+1) - 15n - 6\right] \\ &= \frac{(n+1)}{6}\left[6n^4 + 9n^3 + n^2 - n\right] \\ &= \frac{n(n+1)}{6}\left[6n^3 + 9n^2 + n - 1\right] \\ &= \frac{n(n+1)}{6}(2n+1)(3n^2 + 3n - 1) \end{aligned}$$

and therefore,

$$\sum_{k=1}^{n} k^4 = \frac{1}{30}n(n+1)(2n+1)(3n^2 + 3n - 1).$$

Again, from the binomial theorem we have

$$(k+1)^6 - k^6 = 6k^5 + 15k^4 + 20k^3 + 15k^2 + 6k + 1$$

for each $k = 1, 2, \ldots, n$. Adding these equations, the sum on the left-hand side telescopes and we get

$$(n+1)^6 - 1^6 = 6\sum_{k=1}^{n} k^5 + 15\sum_{k=1}^{n} k^4 + 20\sum_{k=1}^{n} k^3 + 15\sum_{k=1}^{n} k^2 + 6\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1,$$

and using the results above, after simplifying we get

$$\sum_{k=1}^{n} k^5 = \frac{1}{6}n^6 + \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2$$

for $n \geq 1$.

Problem 3. For each of the following

(a) $\sum_{k=1}^{n}(-1)^{k-1}k$

(b) $\sum_{k=1}^{n}(-1)^{k-1}k^2$

(c) $\sum_{k=1}^{n}(-1)^{k-1}k(k-1)$

find a closed form expression valid for $n \geq 1$. Justify your answers, using mathematical induction or otherwise.

SOLUTION:

(a) Note that

$$\begin{aligned} 1 &= 1 \\ 1-2 &= -1 \\ 1-2+3 &= 2 \\ 1-2+3-4 &= -2 \\ 1-2+3-4+5 &= 3 \\ 1-2+3-4+5-6 &= -3 \\ &\vdots \end{aligned}$$

and it appears that

$$\sum_{k=1}^{n}(-1)^{k-1}k = (-1)^{n-1}\left\lfloor \frac{n+1}{2} \right\rfloor \tag{$*$}$$

for $n \geq 1$. We prove this using induction, the base case $n = 1$ has already been shown to hold, suppose that $(*)$ is true for some $n \geq 1$, then

$$\begin{aligned} \sum_{k=1}^{n+1}(-1)^{k-1}k &= \sum_{k=1}^{n}(-1)^{k-1}k + (-1)^n(n+1) \\ &= (-1)^{n-1}\left\lfloor \frac{n+1}{2} \right\rfloor + (-1)^n(n+1) \\ &= (-1)^n\left\{(n+1) - \left\lfloor \frac{n+1}{2} \right\rfloor\right\} \\ &= (-1)^n\left\lfloor \frac{n+2}{2} \right\rfloor, \end{aligned}$$

since

$$\left\lfloor \frac{n+2}{2} \right\rfloor = n+1-\left\lfloor \frac{n+1}{2} \right\rfloor,$$

and $(*)$ is true for $n+1$ also. Therefore, by the principle of mathematical induction $(*)$ is true for all $n \geq 1$.

(b) Note that

$$\begin{aligned}
-1^2 &= -1 \\
-1^2 + 2^2 &= 3 = \frac{3 \cdot 2}{2} \\
-1^2 + 2^2 - 3^2 &= -6 = -\frac{4 \cdot 3}{2} \\
-1^2 + 2^2 - 3^2 + 4^2 &= 10 = \frac{5 \cdot 4}{2} \\
-1^2 + 2^2 - 3^2 + 4^2 - 5^2 &= -15 = -\frac{6 \cdot 5}{2} \\
-1^2 + 2^2 - 3^2 + 4^2 - 5^2 + 6^2 &= 21 = \frac{7 \cdot 6}{2} \\
&\vdots
\end{aligned}$$

and it appears that

$$\sum_{k=1}^{n} (-1)^{k-1} k^2 = (-1)^{n-1} \cdot \frac{n(n+1)}{2} \tag{$**$}$$

for $n \geq 1$. We prove this using induction, the base case $n = 1$ has already been shown to hold, suppose that $(**)$ is true for some $n \geq 1$, then

$$\begin{aligned}
\sum_{k=1}^{n+1} (-1)^{k-1} k^2 &= \sum_{k=1}^{n} (-1)^{k-1} k^2 + (-1)^n (n+1)^2 \\
&= (-1)^{n-1} \cdot \frac{n(n+1)}{2} + (-1)^n (n+1)^2 \\
&= (-1)^n (n+1) \left\{ n + 1 - \frac{n}{2} \right\} \\
&= (-1)^n (n+1) \left\{ \frac{n}{2} + 1 \right\} \\
&= (-1)^n \cdot \frac{(n+1)(n+2)}{2},
\end{aligned}$$

and $(**)$ is true for $n + 1$ also. Therefore, by the principle of mathematical induction $(**)$ is true for all $n \geq 1$.

(c) Note that

$$\begin{aligned}
\sum_{k=1}^{n} (-1)^{k-1} k(k-1) &= \sum_{k=1}^{n} (-1)^{k-1} k^2 - \sum_{k=1}^{n} (-1)^{k-1} k \\
&= (-1)^{n-1} \cdot \frac{n(n+1)}{2} - (-1)^{n-1} \left\lfloor \frac{n+1}{2} \right\rfloor,
\end{aligned}$$

that is,

$$\sum_{k=1}^{n} (-1)^{k-1} k(k-1) = (-1)^{n-1} \left\{ \frac{n(n+1)}{2} - \left\lfloor \frac{n+1}{2} \right\rfloor \right\}$$

for $n \geq 1$.

Problem 4. For each $n \geq 1$, let a_n be the number of ways to group $2n$ people into pairs.

(a) Find a recurrence relation and an initial condition satisfied by the sequence $\{a_n\}_{n\geq 1}$.

(b) Conjecture a value for a_n, and prove your conjecture is true.

SOLUTION:

(a) Select one person x, then there are $2n-1$ choices for x's partner, and there are $2n-2$ people left. The remaining $2n-2$ people can then be paired off in a_{n-1} ways.

Thus, we have $2n-1$ ways to choose the partner for person x, and for each of these, we have a_{n-1} choices for the other pairs, and $\{a_n\}_{n\geq 1}$ satisfies the recurrence relation

$$\begin{aligned} a_n &= (2n-1)a_{n-1}, \quad n \geq 2 \\ a_1 &= 1. \end{aligned}$$

(b) Solving this recurrence relation from the top down, we have

$$\begin{aligned} a_n &= (2n-1)a_{n-1} \\ &= (2n-1)(2n-3)a_{n-2} \\ &= (2n-1)(2n-3)(2n-5)a_{n-3} \\ &\vdots \\ &= (2n-1)(2n-3)(2n-5)\cdots(2n-(2k-1))a_{n-k}. \end{aligned}$$

When $k = n-1$, the recursion stops at a_1, so that

$$\begin{aligned} a_n &= (2n-1)(2n-3)(2n-5)\cdots 5 \cdot 3 \cdot a_1 \\ &= (2n-1)(2n-3)(2n-5)\cdots 5 \cdot 3 \cdot 1, \end{aligned}$$

and therefore

$$a_n = \frac{(2n)!}{2 \cdot 4 \cdot 6 \cdots 2n} = \frac{(2n)!}{2^n\, n!} \qquad (***)$$

for $n \geq 1$.

Suppose that $\{a_n\}_{n\geq 1}$ satisfies the recurrence relation

$$\begin{aligned} a_n &= (2n-1)a_{n-1}, \quad n \geq 2 \\ a_1 &= 1, \end{aligned}$$

we will show by induction that $(***)$ holds for all $n \geq 1$.

Base Case: For $n = 1$, we have $\dfrac{(2 \cdot 1)!}{2^1 \cdot 1!} = 1$, and $a_1 = 1$, so that $(***)$ is true for $n = 1$.

Inductive Step: Assume that $(***)$ is true for some $n \geq 1$, then from the recurrence relation and the inductive hypothesis we have

$$a_{n+1} = [2(n+1)-1]a_n = (2n+1)a_n = (2n+1)\frac{(2n)!}{2^n\, n!} = \frac{(2n+2)!}{2^{n+1}\,(n+1)!},$$

and $(***)$ is also true for $n+1$. By the principle of mathematical induction, $(***)$ is true for all $n \geq 1$.

Problem 5. Show that

$$a_n = \frac{1}{n+1}\binom{2n}{n}$$

is an integer for $n = 1, 2, 3 \ldots$.

SOLUTION: We have

$$\begin{aligned}
\frac{1}{n+1}\binom{2n}{n} &= \frac{(2n)!}{n!(n+1)!} \\
&= \frac{(2n)!}{n!(n+1)!}\left[(n+1)-n\right] \\
&= \frac{(2n)!}{(n!)^2} - \frac{(2n)!}{(n-1)!(n+1)!} \\
&= \binom{2n}{n} - \binom{2n}{n-1},
\end{aligned}$$

and since $\binom{2n}{n}$ and $\binom{2n}{n-1}$ are integers, then

$$\frac{1}{n+1}\binom{2n}{n}$$

is an integer.

Problem 6. A certain computer system considers a string of bits a valid codeword if and only if it contains an even number of $1's$. For example 1 0 0 0 0 1 is a valid codeword, but 1 0 0 1 0 0 1 is not. Let a_n be the number of valid n-bit codewords.

(a) Find a recurrence relation and an initial condition satisfied by a_n.

(b) Given a positive integer N, how many valid codewords of length at most N are there?

SOLUTION:

(a) The number of n-bit valid codewords that end with a 0 is a_{n-1}, since in this case the initial $n-1$ bits must form a valid codeword.

The number of n-bit valid codewords that end with a 1 is $2^{n-1} - a_{n-1}$, since in this case the initial $n-1$ bits must form an invalid codeword, and there are $2^{n-1} - a_{n-1}$ of these.

Therefore, since every n-bit valid codeword must end in 0 or a 1, the number of valid n-bit codewords satisfies the recurrence relation

$$a_n = a_{n-1} + \left(2^{n-1} - a_{n-1}\right) = 2^{n-1}$$

for $n \geq 2$.

The initial condition satisfied by a_n is $a_1 = 1$, since the only 1-bit valid codeword is 0.

(b) The number of valid codewords of length at most N is

$$a_1 + a_2 + a_3 + \cdots + a_N = 1 + 2 + 2^2 + \cdots + 2^{N-1} = \frac{2^N - 1}{2 - 1} = 2^N - 1.$$

Problem 7. A certain basketball team can only sink foul shots and lay-ups, worth 1 and 2 points, respectively. Let a_n denote the number of ways the team can score n points. (Scoring 1 then 2 is considered to be different than scoring 2 then 1). Write down a recurrence relation for a_n with initial conditions for a_0 and a_1, and explain why it holds for all $n \geq 2$. What is the solution to this recurrence relation?

Solution: For each $n \geq 2$, there are exactly two ways to score n points, either the team has scored $n-1$ points previously, and then scores 1 point, or the team has scored $n-2$ points previously, and then scores 2 points. Since these are the only two possibilities, then we must have

$$a_n = a_{n-1} + a_{n-2}$$

for all $n \geq 2$.

If $n = 0$, there is only one way to score this number of points, namely, do not sink any foul shots and do not sink any lay-ups, therefore $a_0 = 1$.

If $n = 1$, there is only one way to score this number of points, namely, sink a foul shot, therefore $a_1 = 1$.

The recurrence relation together with the initial conditions are

$$\begin{aligned} a_n &= a_{n-1} + a_{n-2}, \quad n \geq 2 \\ a_0 &= 1 \\ a_1 &= 1. \end{aligned}$$

We can easily see that this recurrence relation is generating the **Fibonacci numbers**

$$a_0 = F_1 = 1, \quad a_1 = F_2 = 1, \quad a_2 = F_3 = 2, \quad a_3 = F_4 = 3, \quad \ldots$$

and in general, $a_n = F_{n+1}$ for all $n \geq 0$, a fact which can be easily proven by induction.

Let $b_n = a_n - F_{n+1}$ for $n \geq 0$, we will show by induction that $b_n = 0$ for all $n \geq 0$.

Base Case: Note that $b_0 = a_0 - F_1 = 1 - 1 = 0$, $b_1 = a_1 - F_2 = 1 - 1 = 0$, and $b_2 = a_2 - F_3 = 2 - 2 = 0$, and the result is true for $0 \leq k \leq 2$.

Inductive Step: Assume that $b_k = 0$ for $0 \leq k \leq n-1$, then

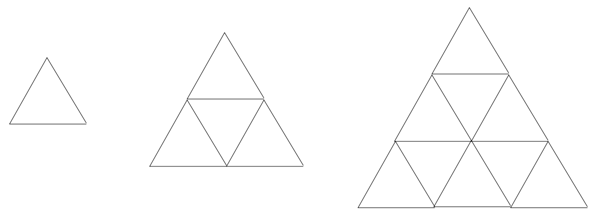
$$b_n = a_n - F_{n+1} = a_{n-1} + a_{n-2} - (F_n + F_{n-1}) = (a_{n-1} - F_n) + (a_{n-2} - F_{n-1}) = b_{n-1} + b_{n-2} = 0,$$

and the result is true for n also.

Therefore, by the principle of strong mathematical induction, $b_n = 0$ for all $n \geq 0$, that is, $a_n = F_{n+1}$ for all $n \geq 0$.

Problem 8. Given a positive integer n, consider an equilateral triangle of side n made up of n^2 equilateral triangles of side 1. Let a_n be the total number of equilateral triangles present for $n \geq 1$.

For example, in figure below,



we have

$$a_1 = 1 \qquad a_2 = 4 + 1 = 5 \qquad a_3 = 9 + 3 + 1 = 13$$

(a) Find a recurrence relation and initial condition satisfied by the sequence $\{a_n\}_{n\geq 1}$.

(b) Conjecture a value for a_n, and prove your conjecture is true.

SOLUTION:

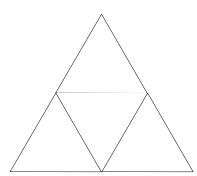
(a) We can count the number of equilateral triangles a_n present when we add another row to an equilateral triangle of side $n - 1$ as follows:

$$a_n = a_{n-1} + U(n) + D(n)$$

where $U(n)$ is the number of new triangles added which are pointing upwards and $D(n)$ equals the number of new triangles added which are pointing downwards.

For example, $a_1 = 1$, and

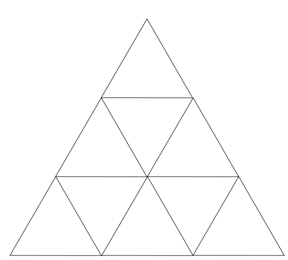
For $n = 2$, we have $U(2) = 2 + 1 = 3$ and $D(2) = 1$, as in the figure,



therefore,

$$a_2 = a_1 + U(2) + D(2) = 1 + 3 + 1 = 5.$$

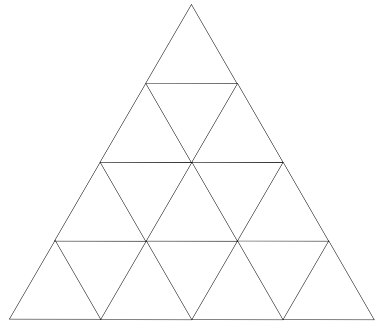
For $n = 3$, we have $U(3) = 3 + 2 + 1 = 6$ and $D(3) = 2 + 0 = 2$, as in the figure,



therefore,

$$a_3 = a_2 + U(3) + D(3) = 5 + 6 + 2 = 13.$$

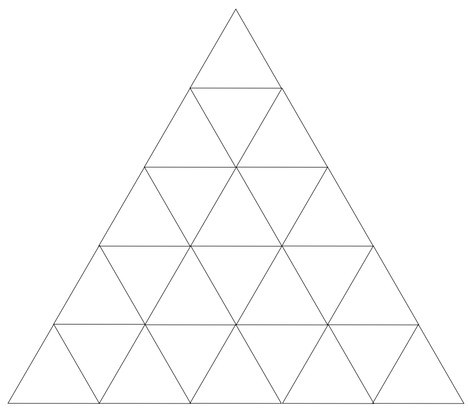
For $n = 4$, we have $U(4) = 4 + 3 + 2 + 1 = 10$ and $D(4) = 3 + 1 = 4$, as in the figure,



therefore,

$$a_4 = a_3 + U(4) + D(4) = 13 + 10 + 4 = 27.$$

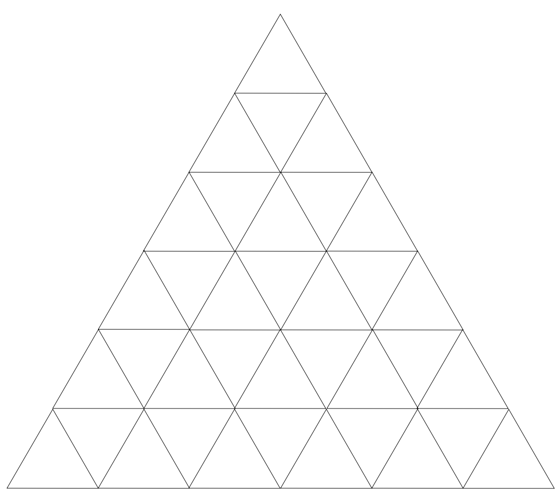
For $n = 5$, we have $U(5) = 5 + 4 + 3 + 2 + 1 = 15$ and $D(5) = 4 + 2 = 6$, as in the figure,



therefore,

$$a_5 = a_4 + U(5) + D(5) = 27 + 15 + 6 = 48.$$

For $n = 6$, we have $U(6) = 6 + 5 + 4 + 3 + 2 + 1 = 21$ and $D(6) = 5 + 3 + 1 = 9$, as in the figure,



therefore,

$$a_6 = a_5 + U(6) + D(6) = 48 + 21 + 9 = 78.$$

In general, for $n \geq 3$, counting the new triangles according to their side lengths, we have

$$U(n) = n + n - 1 + n - 2 + \cdots + 2 + 1 = \frac{n(n+1)}{2},$$

and

$$\begin{aligned} D(n) &= n - 1 + n - 3 + \cdots + n - (2\lfloor n/2 \rfloor - 1) \\ &= n\left\lfloor \frac{n}{2} \right\rfloor - (1 + 3 + \cdots + (2\lfloor n/2 \rfloor - 1) \\ &= n\left\lfloor \frac{n}{2} \right\rfloor - \left\lfloor \frac{n}{2} \right\rfloor^2 \\ &= \left\lfloor \frac{n}{2} \right\rfloor \left(n - \left\lfloor \frac{n}{2} \right\rfloor \right), \end{aligned}$$

that is,

$$D(n) = \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n+1}{2} \right\rfloor$$

since $n - \left\lfloor \frac{n}{2} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor$.

Therefore a_n satisfies the discrete initial value problem

$$\begin{aligned} a_n &= a_{n-1} + \frac{n(n+1)}{2} + \left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n+1}{2} \right\rfloor, \quad n \geq 2 \\ a_1 &= 1. \end{aligned}$$

(b) Using the fact that

$$\left\lfloor \frac{n}{2} \right\rfloor = \frac{2n - (1 - (-1)^n)}{4} \qquad \text{and} \qquad \left\lfloor \frac{n+1}{2} \right\rfloor = \frac{2n + (1 - (-1)^n)}{4},$$

we see that

$$\left\lfloor \frac{n}{2} \right\rfloor \cdot \left\lfloor \frac{n+1}{2} \right\rfloor = \frac{n^2}{4} - \frac{(1 - (-1)^n)}{8},$$

and the difference equation becomes

$$a_n = a_{n-1} + \frac{3n^2}{4} + \frac{n}{2} - \frac{(1 - (-1)^n)}{8}$$

for $n \geq 1$. Therefore

$$\begin{aligned} a_n - a_1 &= \sum_{k=2}^{n} (a_k - a_{k-1}) \\ &= \frac{3}{4} \sum_{k=2}^{n} k^2 + \frac{1}{2} \sum_{k=2}^{n} k - \frac{1}{8} \sum_{k=2}^{n} (1 - (-1)^k) \\ &= \frac{3}{4} \left(\frac{n(n+1)(2n+1)}{6} - 1 \right) + \frac{1}{2} \left(\frac{n(n+1)}{2} - 1 \right) - \frac{1}{8} \sum_{k=2}^{n} (1 - (-1)^k) \\ &= \frac{n(n+1)(2n+1)}{8} - \frac{3}{4} + \frac{n(n+1)}{4} - \frac{1}{2} - \frac{1}{8} \sum_{k=2}^{n} (1 - (-1)^k), \end{aligned}$$

and since $a_1 = 1$, after simpifying, we get

$$a_n = \frac{n(n+1)(2n+3)}{8} - \frac{1}{4} - \frac{1}{8}\sum_{k=2}^{n}(1-(-1)^k).$$

Now,

$$\frac{1}{8}\sum_{k=2}^{n}(1-(-1)^k) = \frac{n-1}{8} - \frac{(1+(-1)^n}{16} = \frac{n}{8} - \frac{3}{16} - \frac{(-1)^n}{16},$$

so that

$$\begin{aligned} a_n &= \frac{n(n+1)(2n+3)}{8} - \frac{1}{4} - \frac{n}{8} + \frac{3}{16} + \frac{(-1)^n}{16} \\ &= \frac{n(n+1)(2n+3)}{8} - \frac{n}{8} - \frac{1-(-1)^n}{16} \\ &= \frac{n}{8}\left[(n+1)(2n+3)-1\right] - \frac{1-(-1)^n}{16} \\ &= \frac{n}{8}\left[2n^2+5n+2\right] - \frac{1-(-1)^n}{16} \\ &= \frac{n(n+2)(2n+1)}{8} - \frac{1-(-1)^n}{16}, \end{aligned}$$

and therefore

$$a_n = \frac{n(n+2)(2n+1)}{8} - \frac{1-(-1)^n}{16}$$

for $n \geq 1$. An easy induction proof shows that this is indeed the solution to the recurrence relation satisfying the initial condition.