math 222

Solutions to Assignment 2

Problem 1. Consider the following list of binary numbers (it goes on forever):

$$\begin{array}{c}
1\ \ 1 \\
1\ \ 0\ \ 1 \\
1\ \ 0\ \ 0\ \ 1 \\
1\ \ 0\ \ 0\ \ 0\ \ 1 \\
1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1 \\
1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1 \\
1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1 \\
1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1 \\
1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 1 \\
.\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .\ \ .
\end{array}$$

Which of these integers is divisible by 3 ? Justify your answer.

SOLUTION: Note that the binary numbers in the list represent the integers

$$2^n + 1$$

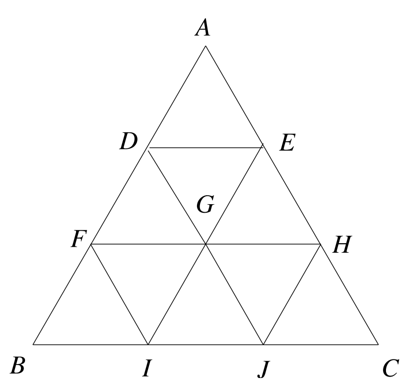
for $n = 1, 2, 3, \ldots$, we will show that $3 \mid 2^n + 1$ if and only if $n \equiv 1 \pmod 2$, that is, if and only if the positive integer n is odd.

Since $2 \equiv -1 \pmod 3$, then $2^n \equiv (-1)^n \pmod 3$, and therefore

$$2^n + 1 \equiv 0 \pmod 3$$

if and only if n is odd.

Problem 2. Let the triangle ABC be equilateral with $AB = 3$. Show that if we select 10 points in the interior of this triangle, there must be at least two whose distance apart is less than or equal to 1.



Solution: If we divide the triangle into 9 nonoverlapping regions as shown in the figure above, then it is clear from the pigeonhole principle that at least two of the 10 points lie in the same region, and so are at a distance less than or equal to 1 from each other.

We can actually do a little better. We partition the interior of the triangle ABC into 9 *pairwise disjoint* regions as shown in the figure above, where each line segment interior to triangle ABC belongs to exactly one of the two triangles it borders on, and the point G belongs to exactly one of the regions.

Now, if we select 10 points from the interior of triangle ABC, then from the pigeon-hole principle at least two of these points are in the same region and so are at a distance less than 1 from each other.

Problem 3. The "two-out-of-five" code consists of all possible binary words of length 5 containing exactly two 1's.

(a) List all of the code words.

(b) What is the minimum Hamming distance between code words?

(c) How many errors can the code detect?

(d) How many errors can the code correct?

Solution:

(a) The number of distinct code words is $\binom{5}{2} = \frac{5 \cdot 4}{2 \cdot 1} = 10$, and we can list them as follows

1	1	0	0	0
1	0	1	0	0
1	0	0	1	0
1	0	0	0	1
0	1	1	0	0
0	1	0	1	0
0	1	0	0	1
0	0	1	1	0
0	0	1	0	1
0	0	0	1	1

(b) Let a and b be two distinct code words. If the number of positions where the two 1's overlap is 0, for example, $a = 0\,1\,1\,0\,0$ and $b = 0\,0\,0\,1\,1$, then the Hamming distance is $d(a, b) = 4$.

If the number of positions where the two 1's overlap is 1, for example, $a = 0\,1\,1\,0\,0$ and $b = 1\,1\,0\,0\,0$, then the Hamming distance is $d(a, b) = 2$.

Since these are the only two possibilities, then the minimum Hamming distance between any two code words in the "two-out-of-five" code is 2.

(c) Since the minimum Hamming distance between any two code words is 2, this code can detect up to one error.

(d) In order to correct a single error, the minimum Hamming distance between any two code words has to be at least 3, therefore this code cannot correct any errors.

Problem 4. The following message is received using the 15-digit Hamming Code. Correct the number if it is not correct.

a	a	a	a		a	a	a				a			
b	b	b		b	b			b	b			b		
c	c		c	c		c		c		c			c	
d		d	d	d			d		d	d				d
1	1	1	1	0	0	0	0	0	0	0	1	1	1	1

Solution: Only the parity check bit in the column labeled by a is incorrect, and therefore an error occurred in the column labelled by the subset $\{a\}$. The corrected codeword is

$$1 \quad 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \quad 0 \,\big|\, 0 \quad 1 \quad 1 \quad 1$$

Problem 5. There are three curtains — labelled A, B, and C. Behind one of them is a brand new BMW, behind each of the others is a goat. Eleven people know what is behind each curtain. You may ask each person one question, which has to have a yes or no answer. Unfortunately, three of them may lie. Devise a set of questions to ask them, and explain clearly why you will be able to tell from the answers where the BMW is.

Solution: Since up to three of the people may lie, we need code words such that the Hamming distance between any two of them is at least 7. For example, the code words could be

			A				B				C	
a	=	1	1	1	1	0	0	0	0	0	0	0
b	=	0	0	0	0	1	1	1	1	0	0	0
c	=	0	0	0	0	0	0	0	0	1	1	1

Thus, the first 4 questions you ask are: "Is the BMW behind curtain A?"

The next 4 questions you ask are: "Is the BMW behind curtain B?"

You then ask the remaining 3 questions: "Is the BMW behind curtain C?"

Problem 6. A message is divided into 6 parts and copies are made. They are dispatched to an agent behind enemy lines by 6 couriers, 2 of whom may be captured.

(a) What is the minimum number of each part that must be carried.

(b) Explain why no courier can carry four parts.

(c) Devise a scheme by which the agent will get at least one copy of each part, while the enemy cannot get copies of all parts.

Solution:

(a) At least 3 copies of each part must be carried in order to ensure that a complete message gets through since up to 2 parts may be captured, so the minimum number of parts that must be carried is $18 = 3 \cdot 6$.

(b) If one courier carried 4 parts, say A, B, C, D, then parts E and F cannot be carried together. This means that the 3 copies of E and F must each be carried by different couriers, requiring 7 couriers in all.

(c) Parts (a) and (b) imply that each of the 6 couriers must carry exactly 3 parts. Now, if one courier carries {A, B, C} then no other courier can carry {D, E, F}. This leads to 10 partitions of {A, B, C, D, E, F} into 3 and 3 such that only one side of the partition can be carried by the couriers:

(A B C, D E F) (A B D, C E F) (A B E, C D F) (A B F, C D E) (A C D, B E F)
(A C E, B D F) (A C F, B D E) (A D E, B C F) (A D F, B C E) (A E F, B C D).

Since three of each type must be carried, we can start by having two couriers carrying AB, two carrying CD and two carrying EF. Each will have to carry a third part, and an examination of the partitions shows that the following is a solution:

Courier #1:	ABC
Courier #2:	ABD
Courier #3:	CDE
Courier #4:	CDF
Courier #5:	EFA
Courier #6:	EFB.

Problem 7. You have two parents, four grandparents, eight great-grandparents, and so on $\cdots$.

(a) If all of your ancestors were distinct, what would be the total number of your ancestors for the past 40 generations (counting your parents' generation as number one)?

(b) Assuming that each generation represents 30 years, how long is 40 generations?

(c) The total number of people who have ever lived is approximately 10 billion, which equals 10^{10} people. Compare this fact with the answer to part (a). What do you deduce?

SOLUTION:

(a) It is easy to see that if all of your ancestors were distinct, the number of ancestors in generation number k would be 2^k for $k > 1$. Thus the total number of your ancestors for the past 40 generations would be

$$2 + 2^2 + \cdots + 2^{40} = 2^{41} - 2.$$

(b) If each generation represents 30 years, then 40 generations would represent $40(30) = 1200$ years.

(c) Now, $2^{10} = 1024$, so

$$2^{41} - 2 > 2^{40} = (1024)^4 > (1000)^4 = 10^{12}$$

It seems that you have more ancestors than people who have ever lived. In order to avoid the unthinkable, that is, that not all of your ancestors were distinct, the only reasonable deduction I can make is that we have been invaded by aliens.

Problem 8. Twenty girls are sitting at a round table. In front of each are three lights, one green, one red, and one yellow. Each girl is wearing a green hat or a red hat, and each can see all of the hats except her own. The girls are perfect logicians, and they are going to play a game. The object of the game is for each girl to turn on a light that matches the colour of her hat. The game will be played in several rounds. There is a referee who gives a signal to start the round, and at the signal each girl must turn on one of the lights. A girl who does not know the colour of her hat must turn on the yellow light. Before the game begins, the referee tells the girls that at least one of them is wearing a green hat. In fact, ten have green hats and ten have red hats. Explain what happens in each round of the game until all of the girls have determined their hat colours.

Each girl can actually see that at least one of them is wearing a green hat. So is it necessary for the referee to give them that information?

Solution The game lasts 11 rounds. In rounds one through nine, all of the girls turn on yellow lights. In round 10 all of the girls with green hats turn on green lights and the rest turn on yellow lights. In round eleven each girl turns on the light that matches her hat colour.

We will prove the following using mathematical induction on the number of girls wearing green hats. Note that an inductive proof was not required for the assignment, you only needed to explain what was happening at the end of each round.

We assume that there are a fixed number of girls say n, and at least one of them is wearing a green hat.

We will show that if there are k girls wearing a green hat, then the following are true:

- If, prior to the signal for the k^{th} round, a girl sees k or more green hats and no one has turned on a green or red light, then she will not be able to deduce the color of her hat, and at the signal for the k^{th} round she will turn on her yellow light.
- If, prior to the signal for the k^{th} round, a girl sees $k-1$ green hats and no one has turned on a green or red light, then she will deduce that she has a green hat, and at the signal for the k^{th} round she will turn on her green light.
- If, prior to the signal for the k^{th} round, a girl sees $k-1$ green hats and the green lights have been turned on, then she will deduce that she has a red hat, and at the signal for the k^{th} round she will turn on her red light.

Or, more simply put, for the first $k-1$ rounds at the signal for the round, all the girls turn on the yellow lights. At the signal for the k^{th} round, all k girls wearing green hats turn on the green lights, and the remaining girls turn on the yellow lights. Finally, at the signal for the $k+1^{\text{st}}$ round, all the girls wearing green hats turn on the green lights and all the girls wearing red hats turn on the red lights, thus, the game lasts $k+1$ rounds.

Base Case: For $k=1$, if there is only one green hat, the girl wearing the green hat does not see any other green hat, and since the referee stated there was at least one green hat, she deduces that her hat is green. Any girl wearing a red hat can see at least one green hat, and so cannot deduce anything about the color of her own hat. Thus, at the end of the first round: the girl wearing a green hat turns on a green light, while all the other girls turn on a yellow light.

Prior to the second round, any girl with a red hat sees exactly one green hat and reasons as follows: if my hat were green, the girl with the green hat would not be able to deduce the color of her own hat, and would have turned on the yellow light at the end of round 1; therefore, my hat must be red. Thus, at the end of the second round: the girl wearing a green hat turns on a green light, and all the girls with red hats turn on a red light, and the game ends after 2 rounds.

Inductive Step: Suppose that $1 \leq k < n$, and assume that there are $k+1$ green hats, and that for the first k rounds, all of the girls turned on a yellow light. If you have a green hat on your head, you see k other green hats, and if your hat were red, then by the inductive hypothesis, all of the girls wearing green hats should have turned on a green light after round k, but they didn't. Therefore, your hat must be green. Also, from the inductive hypothesis any girl wearing a red hat would not be able to deduce that fact until after the $k+1^{\text{st}}$ round. Thus, at the end of the $k+1^{\text{st}}$ round: all the girls wearing a green hat turn on the green light, and all the girls wearing a red hat turn on a yellow light.

Prior to the $k+2^{\text{nd}}$ round, any girl with a red hat sees exactly $k+1$ green hats and reasons as follow: if my hat were green, the girls with the green hats would not be able to deduce the color of their own hats, and would have turned on the yellow lights at the end of round $k+1$; therefore my hat must be red. Thus, at the end of the $k+2^{\text{nd}}$ round: all the girls wearing green hats turn on a green light, and all the girls wearing red hats turn on a red light, and the game ends after $k+2$ rounds.

Note: The base case cannot be established until they are informed of the fact that at least one is wearing a green hat. Suppose the referee does not issue the statement that at least one of them is wearing a green hat. See what happens if only one girl is wearing a green hat. In round one, none of the girls with red hats can deduce their hat colour, and neither can the girl with the green hat. In round two, the situation is the same.

Suppose there were two girls with green hats and that you were one of them. In round one no one can deduce the colour of their hat, so everyone's response is exactly the same regardless of the colour of your hat. In round 2, although you see a girl with a green hat, you have gathered no more information than you did in round 1, so you will not be able to determine the colour of your hat. So the inductive reasoning process collapses without that seemingly extraneous statement by the referee.