math22

Solutions to Assignment 1

Problem 1. Two grade six classes were going on a field trip to a museum. They were being transported by two buses each of which had 34 seats. It so happened that there were 30 boys and 34 girls, and so they put all the boys on one bus and the girls on the other bus. The buses had to stop for a few minutes, and at that time several boys sneaked onto the girls' bus. But the girls' bus driver noticed that there were 10 too many on the bus, so he sent 10 children (boys and girls) back to the boys' bus. After this was done, were there more boys on the girls' bus than girls on the boys' bus? Or vice versa?

Solution: Suppose that when all the moving around was over and done with, there were x boys on the girls' bus. Since there were 30 children on the boys' bus and 34 children on the girls' bus, the x boys on the girls' bus replaced x girls. Where were these x girls? They were on the boys' bus. Therefore there were the same number of boys on the girls' bus as there were girls on the boys' bus.

Problem 2. There were 5 children in the Emergency Room and between them they had stuck a total of 76 beans up their noses. Show that there must be three children with a combined total of 45 or more beans up their noses.

Solution: Suppose there do not exist 3 children with a combined total of 45 or more beans up their noses. If we label the children A, B, C, D, and E and a, b, c, d, and e are the number of beans in each child's nose, respectively, then we have $x+y+z \leq 44$ for any triple $x, y, z \in \{a, b, c, d, e\}$. There are $\binom{5}{3} = 10$ ways to choose the triplet x, y, z, so that we get 10 inequalities like this. Namely,

$$\begin{aligned}
a+b+c &\leq 44\\
a+b+d &\leq 44\\
a+b+e &\leq 44\\
a+c+d &\leq 44\\
a+c+e &\leq 44\\
a+d+e &\leq 44\\
b+c+d &\leq 44\\
b+c+e &\leq 44\\
b+d+e &\leq 44\\
c+d+e &\leq 44
\end{aligned}$$

Each of the numbers a, b, c, d, e appears in exactly $\binom{4}{2} = 6$ of the expressions on the left hand side of these 10 inequalities.

If we add these 10 inequalities, on the left hand side we get $6 \cdot (a+b+c+d+e)$, while on the right hand side we get $10 \cdot 44$, that is, $6 \cdot (a+b+c+d+e) \leq 10 \cdot 44$, so that $456 = 6 \cdot 76 \leq 10 \cdot 44 = 440$.

Which is obviously a contradiction. Therefore, our original assumption must have been incorrect, and there must be three children with a combined total of 45 or more beans up their noses.

Problem 3. Dr. Ecco concluded that the aliens from whom the message was received also had an "alphabet" consisting of 26 distinct characters. Another moment's thought convinced him that the code was a linear code with encoding function $E(x) \equiv 5x \pmod{26}$. Your mission, should you decide to accept it, is to help Dr. Ecco find the decoding function $D(x)$, that is, find an integer a such that

$$D(x) \equiv ax \pmod{26},$$

and decode the following message from the aliens: `IH FUAN, KADD JSIU`

SOLUTION: Since $5 \cdot 21 \equiv 105 \equiv 1 \pmod{26}$, then the decoding function is

$$D(x) \equiv 21x \pmod{26},$$

for $x = 0, 1, 2 \ldots, 25$.

Decoding the message, we have

I	$\leftrightarrow$	8	$D(8) \equiv 21 \cdot 8 \equiv 168 \equiv 12 \pmod{26}$	12	$\leftrightarrow$	M
H	$\leftrightarrow$	7	$D(7) \equiv 21 \cdot 7 \equiv 147 \equiv 17 \pmod{26}$	17	$\leftrightarrow$	R
F	$\leftrightarrow$	5	$D(5) \equiv 21 \cdot 5 \equiv 105 \equiv 1 \pmod{26}$	1	$\leftrightarrow$	B
U	$\leftrightarrow$	20	$D(20) \equiv 21 \cdot 20 \equiv 420 \equiv 4 \pmod{26}$	4	$\leftrightarrow$	E
A	$\leftrightarrow$	0	$D(0) \equiv 21 \cdot 0 \equiv 0 \pmod{26}$	0	$\leftrightarrow$	A
N	$\leftrightarrow$	13	$D(13) \equiv 21 \cdot 13 \equiv 273 \equiv 13 \pmod{26}$	13	$\leftrightarrow$	N
K	$\leftrightarrow$	10	$D(10) \equiv 21 \cdot 10 \equiv 210 \equiv 2 \pmod{26}$	2	$\leftrightarrow$	C
A	$\leftrightarrow$	0	$D(0) \equiv 21 \cdot 0 \equiv 0 \pmod{26}$	0	$\leftrightarrow$	A
D	$\leftrightarrow$	3	$D(3) \equiv 21 \cdot 3 \equiv 63 \equiv 11 \pmod{26}$	11	$\leftrightarrow$	L
D	$\leftrightarrow$	3	$D(3) \equiv 21 \cdot 3 \equiv 63 \equiv 11 \pmod{26}$	11	$\leftrightarrow$	L
J	$\leftrightarrow$	9	$D(9) \equiv 21 \cdot 9 \equiv 189 \equiv 7 \pmod{26}$	7	$\leftrightarrow$	H
S	$\leftrightarrow$	18	$D(18) \equiv 21 \cdot 18 \equiv 378 \equiv 14 \pmod{26}$	14	$\leftrightarrow$	O
I	$\leftrightarrow$	8	$D(8) \equiv 21 \cdot 8 \equiv 168 \equiv 12 \pmod{26}$	12	$\leftrightarrow$	M
U	$\leftrightarrow$	20	$D(20) \equiv 21 \cdot 20 \equiv 420 \equiv 4 \pmod{26}$	4	$\leftrightarrow$	E

The message reads: `MR BEAN, CALL HOME`

Problem 4. Decrypt the following passage which uses a mono-alphabetic code:

E TSWA-EJB-LEFA MAHELSKJNRSP XSLR UKZM JASTRDKZM SN ANNAJLSEH LK LRA QZHH
AJGKUIAJL KQ UKZM TEMBAJ. JAWAM CMSLSCSNA RSN CMAELSWA AQQKMLN, KM SJN-
SNL KJ TSWSJT RSI ZJNKHSCSLAB EBWSCA. XRAJ RA SN SJ LMKZDHA, DA PMAPEMAB LK
RAHP KZL KQ CKZMNA, DZL UKZ NRKZHB MANPACL RSN PMSWECU EL EHH LSIAN. SQ RSN
EJSIEH NRKZHB NLMEU, HAL RSI FJKX DU EHH IAEJN, DZL BKJ'L DA WSJBSCLSWA. EWKSB
CKJNLEJL DKMMKXSJT. E RKZNARKHBAM SN AJLSLHAB LK MAIKWA EJU KDGACL LREL
AJCMKECRAN KJ RSN PMKPAMLU, DZL SL IZNL DA MALZMJAB LK LRA MSTRLQZH KXJAM.

The frequency table is given below:

A	B	C	D	E	F	G	H	I	J	K	L	M
48	13	13	11	23	2	2	18	8	28	35	40	27

N	O	P	Q	R	S	T	U	V	W	X	Y	Z
27	0	8	7	21	39	7	9	0	10	5	0	17

Solution: Based on the frequency table and word patterns, the message is decrypted as follows.

> *A give-and-take relationship with your neighbour is essential to the full enjoyment of your garden. Never criticise his creative efforts, or insist on giving him unsolicited advice. When he is in trouble, be prepared to help out of course, but you should respect his privacy at all times. If his animal should stray, let him know by all means, but don't be vindictive. Avoid constant borrowing. A householder is entitled to remove any object that encroaches on his property, but it must be returned to the rightful owner.*

Problem 5. The inspectors of fair trading found that a wholesaler of golfing equipment was swindling his retailers by including one box of substandard golf balls to every nine boxes of top grade balls he sold them. Each box contained 6 golf balls, and the external appearance of all the balls was identical. However, the substandard balls were each 1 gram too light. The retailers were informed of this discrepancy. The boxes all arrived in packs of ten, each with one substandard box — but which one?

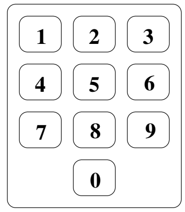
Phoebe Fivewood, the professional at a prestigious golf course, had just taken delivery of a large order and needed to identify the defective ones quickly. She soon found a way to do this using a pair of scales (not pan balances) which required only one weighing on each scale for each batch of ten boxes. How did she do it? Note that she did not need to know what a golf ball should weigh.

Solution: Pheobe Fivewood puts the following balls on scale #1:
one ball from package 1, two from package 2, three from package 3, four from package 4, and five from package 5.

On scale #2 she puts:
one ball from package 6, two from package 7, three from package 8, four from package 9, and five from package 10.

She notes the difference in weight on both scales, and this tells her what package has the defective product, (For example if scale #2 records a weight that is 3 grams less than scale #1, then package 8 is the defective one.)

Problem 6. The telephone numbers in a town run from 00000 to 99999. A common error in dialing on a standard keypad is to punch in a digit **vertically** adjacent to the intended one. So, on a standard dialing keypad, 6 could be erroneously entered as 3 or 9 (but not as 2, 5 or 8).



Assuming that no other kinds of errors are made, how should a sixth digit be added to each telephone number so that no wrong numbers will be reached because of a dialing error?

Solution: Again, there are many acceptable solutions. Given the number $abcde$, one solution is to choose the sixth digit f so that

$$a+b+c+d+e+f \equiv 0 \pmod{10},$$

in other words, so that $a+b+c+d+e+f$ is divisible by 10.

Note that if an error is made then the difference in the two digits is either 3 or 8.

Suppose that in $abcdef$ a digit x gets changed to x', for example, suppose that $x = b$. If the error is not detected, then

$$\begin{aligned} & a+x+c+d+e+f \equiv 0 \pmod{10} \\ \text{and} \qquad & a+x'+c+d+e+f \equiv 0 \pmod{10}\,, \\ \text{and so} \qquad & x-x' \equiv 0 \pmod{10} \end{aligned}$$

However, $x - x' \equiv 0 \pmod{10}$ if and only if $x - x'$ is a multiple of 10, and since x and x' are between 0 and 9, then $-9 \le x - x' \le 9$, but the only multiple of 10 between -9 and 9 is $0 = 0 \cdot 10$. Therefore, if the error is not detected, then $x = x'$, or equivalently, if x is changed to x' and x and x' are different, then the proposed scheme will always catch the error.

Problem 7. There are four vials labeled A, B, C and D. Exactly two vials contain a deadly virus and cannot be opened. The other two vials contain vaccines.
There is a device that can test exactly one of the vials, and it will flash a red light if the vial contains the virus. The device can be used eleven times. It is not entirely reliable, and it may give false results up to two times out of the eleven.

Professor Scarlet devises a method to find the vials containing the virus: Test each of vials A, B and C three times and vial D twice and record the results. Represent these results as strings of 0's and 1's using 0 for a red light and 1 for no light. This produces six code words which are eleven bits long.

(a) List the code words and the Hamming distance between each pair of code words.

(b) When the test was run the results were as follows:

$$\text{vial } A: \quad 1 \quad 1 \quad 0 \qquad \text{vial } B: \quad 0 \quad 0 \quad 0 \qquad \text{vial } C: \quad 0 \quad 0 \quad 1 \qquad \text{vial } D: \quad 1 \quad 1$$

Which two vials contain the virus?

(c) Dr. Ecco claims that there is no need to test vial D, you only need to test each of the others three times. Is he correct?

SOLUTION:

(a) If no errors were made, the output would be

	virus in A and B			virus in A and C			virus in A and D			virus in B and C			virus in B and D			virus in C and D		
A:	0	0	0	0	0	0	0	0	0	1	1	1	1	1	1	1	1	1
B:	0	0	0	1	1	1	1	1	1	0	0	0	0	0	0	1	1	1
C:	1	1	1	0	0	0	1	1	1	0	0	0	1	1	1	0	0	0
D:	1	1		1	1		0	0		1	1		0	0		0	0	

The six codewords are the 0's and 1's in the six columns of the table, and can be listed explicitly as follows, here c_{xy} represents the results of the tests of A, B, C, D if vials X and Y contain the virus.

$$\begin{aligned}
c_{ab} &= 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1 \\
c_{ac} &= 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1 \\
c_{ad} &= 0\ 0\ 0\ 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0 \\
c_{bc} &= 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0\ 0\ 1\ 1 \\
c_{bd} &= 1\ 1\ 1\ 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0 \\
c_{cd} &= 1\ 1\ 1\ 1\ 1\ 1\ 0\ 0\ 0\ 0\ 0
\end{aligned}$$

The Hamming distances between pairs of code words are given below.

$$\begin{array}{lllll}
d(c_{ab}, c_{ac}) = 6 & d(c_{ab}, c_{ad}) = 5 & d(c_{ab}, c_{bc}) = 6 & d(c_{ab}, c_{bd}) = 5 & d(c_{ab}, c_{cd}) = 11 \\
d(c_{ac}, c_{ad}) = 5 & d(c_{ac}, c_{bc}) = 6 & d(c_{ac}, c_{bd}) = 11 & d(c_{ac}, c_{cd}) = 5 & \\
d(c_{ad}, c_{bc}) = 11 & d(c_{ad}, c_{bd}) = 6 & d(c_{ad}, c_{cd}) = 6 & & \\
d(c_{bc}, c_{bd}) = 5 & d(c_{bc}, c_{cd}) = 5 & & & \\
d(c_{bd}, c_{cd}) = 6 & & & &
\end{array}$$

(b) Since the minimum Hamming distance between any two code words is at least 5, we can correct up to 2 errors using the nearest neighbour decoding scheme.

If after the tests were run, the results were

$$\text{vial } A: \quad 1\ \ 1\ \ 0 \qquad \text{vial } B: \quad 0\ \ 0\ \ 0 \qquad \text{vial } C: \quad 0\ \ 0\ \ 1 \qquad \text{vial } D: \quad 1\ \ 1$$

since the Hamming distance between this result and the code word c_{bc} is 2, then the correct code word is c_{bc}, and the virus is in vials B and C.

(c) It was not an easy matter for Dr. Ecco to admit that he was wrong, but Professor Scarlet persisted and finally convinced him of the truth. If we do not test D, but test each of A, B and C three times, and if no errors are made then the correct results are

	virus in A and B			virus in A and C			virus in B and C		
A:	0	0	0	0	0	0	1	1	1
B:	0	0	0	1	1	1	0	0	0
C:	1	1	1	0	0	0	0	0	0

However, if the result 0 1 1 1 1 1 0 0 0 is obtained, then we can not determine whether the virus is in vials A and C, or whether the virus is in vials C and D.

If two errors were made, then the correct code word is $c_{ac} = 0\ 0\ 0\ 1\ 1\ 1\ 0\ 0\ 0$, and this indicates that the virus is in vials A and C.

On the other hand, if only one error were made, the result obtained should have been 1 1 1 1 1 1 0 0 0, which indicates that the virus is in vials C and D.

Problem 8. Does there exist a single-error-correcting code with three code words each of which consists of three bits? Explain your answer.

Solution: No such code exists.

Suppose that c_1, c_2 and c_3 are three code words in a single-error-correcting code, where each of c_1, c_2 and c_3 are 3 bits long. In order to correct a single error, the Hamming distance between any two code words must be at least 3. We may assume that

$$c_1 = 0\ 0\ 0 \qquad \text{and} \qquad c_2 = 1\ 1\ 1.$$

If c_3 is different from both c_1 and c_2, it must contain at least one 0 and at least one 1, and since the third bit in c_3 must be a 0 or a 1, then either c_3 is a Hamming distance 1 from c_1 or c_3 is a Hamming distance 1 from c_2, respec tively. Therefore, there is no 3-bit code word at a Hamming distance 3 from both c_1 and c_2.