math 222

SOLUTIONS TO SAMPLE FINAL EXAMINATION

Instructors: I. E. Leonard — Time: 2 Hours

1. Find a closed form expression for the following sequence

$$a_n = 1^2 \cdot n + 2^2 \cdot (n-1) + 3^2 \cdot (n-2) + \cdots + (n-1)^2 \cdot 2 + n^2 \cdot 1$$

for $n = 1, 2, \ldots$.

Hint: $\displaystyle\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and $\displaystyle\sum_{k=1}^{n} k^3 = \frac{n^2(n+1)^2}{4}$.

SOLUTION: For $n \geq 1$, we have

$$\begin{aligned} a_n &= \sum_{k=1}^{n} k^2(n-k+1) \\ &= (n+1)\sum_{k=1}^{n} k^2 - \sum_{k=1}^{n} k^3 \\ &= \frac{n(n+2)^2(2n+1)}{6} - \frac{n^2(n+1)^2}{4} \\ &= \frac{n(n+1)^2}{12}\left[4n+2-3n\right], \end{aligned}$$

so that

$$a_n = \frac{n(n+1)^2(n+2)}{12}$$

for all $n \geq 1$.

2. A certain country uses gold coins called **buckazoids**. The coins weigh the same as their denominations, that is, a 1-buckazoid coin weighs 1 gram, a 2-buckazoid coin weighs 2 grams, and a 3-buckazoid coin weighs 3 grams, etc.

Someone has given you one 1-buckazoid coin, one 2-buckazoid coin, one 3-buckazoid coin, and one 5-buckazoid coin. One of the four coins is fake, and it weighs too much. The other three coins are real and are the correct weight.

(a) How can you find the fake coin by using only a pan balance? You have no other weights except for the coins themselves. Try to use as few weighings as possible.

(b) How can you find the fake coin if all you know is that the fake coin is the wrong weight but you don't know whether it is too heavy or too light? Again, try to use as few weighings as possible.

SOLUTION: Label the coins as 1, 2, 3, and 5. In each case it can be done with two weighings:

(a) Compare 5 vs (3 and 2). If they balance, the counterfeit is 1; otherwise, 1 is real.

    If 5 is heavier it is the counterfeit. If is lighter, compare 3 vs (2 and 1). Since 1 is real, this reveals the counterfeit.

(b) Compare 5 vs (3 and 2). If they balance, the counterfeit is 1; otherwise, 1 is real.

    If 5 is heavy compare 3 vs (2 and 1). If this balances, 5 is the counterfeit, otherwise 5 is real and the counterfeit is 3 or 2 and is light.

    If 3 is heavy, then 2 is the counterfeit. If 3 is light, then it is the counterfeit.

3. The **International Standard Serial Number (ISSN)** applies to publications which are issued in successive parts and are intended to be continued indefinitely. Examples are newspapers, magazines, annual yearbooks and academic journals. The ISSN is an 8-digit codeword with decimal digits, and the last digit is introduced as a check-digit.

   If the first seven digits are $a\,b\,c\,d\,e\,f\,g$, then the eighth digit h is chosen so that

$$8a + 7b + 6c + 5d + 4e + 3f + 2g + h \equiv 0 \text{ (mod 11)}.$$

   Spaces and dashes are ignored, and if h has to be ten, an X is used instead.

   (a) Determine the check-digit for the *Journal of Classical Music*, for which the first seven digits of the ISSN are

$$0317 - 847\square$$

   that is, find the digit that goes in the square.

   (b) Show that this code detects all single errors.

SOLUTION:

(a) $8(0) + 7(3) + 6(1) + 5(7) + 4(8) + 3(4) + 2(7) + h = 120 + h \equiv 0 \pmod{11} \implies h = 1.$

(b) Suppose that the n^{th} digit from the right gets changed from x to y. Assume that in both cases, the congruence is satisfied. Then subtracting one from the other gives

$$nx - ny \equiv 0 \pmod{11}.$$

    This means that either n or $x - y$ is a multiple of 11, and since n is not, then $x - y$ must be. Since $-8 \le x - y \le 8$, this can only happen if $x - y = 0$. But then $x = y$ and no error would have been made. So if an error is made, the congruence cannot be satisfied.

4. The Department of Mathematics at Podunk University has a decision-making committee with a place for one student to serve as a member. Three students, Martha (a freshman), Henry (a junior), and Ralph (a graduate student), are selected as a committee of three to determine which student should serve on this committee. They decide that the freshman, sophomore, junior, and senior classes, and the graduate students, should each nominate one student A, B,C, D and E, respectively, for membership on this faculty-student committee. The voting blocks or preference charts for each of Martha, Henry, and Ralph are

Martha	*Henry*	*Ralph*
C	A	E
D	B	D
E	C	B
A	D	A
B	E	C

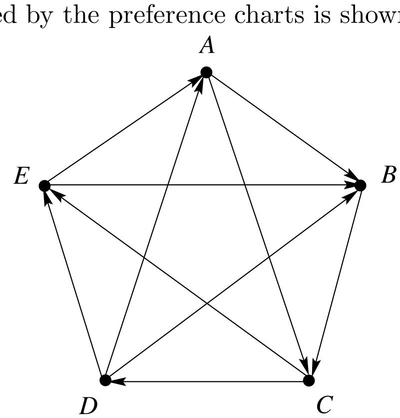
The preferences are listed in descending order. This means, for example, that Martha prefers C to D to E to A to B.

(a) Draw the tournament which is generated by these preference charts.

(b) Ralph wants the graduate student to win. What sequence of one-on-one elections should he use?

SOLUTION:

(a) The tournament generated by the preference charts is shown below.



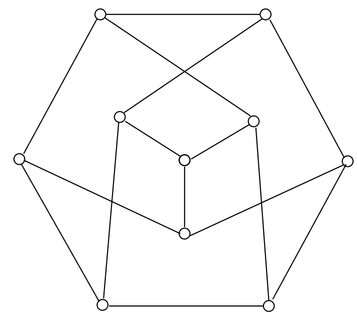
(b) Since the tournament contains the directed Hamiltonian path

$$E \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow D$$

Ralph can arrange for the graduate student to win the election by holding 4 one-on-one elections as follows:

Election 1:	C vs D
Election 2:	winner of 1 vs B
Election 3:	winner of 2 vs A
Election 4:	winner of 3 vs E

5. Let P be the Petersen graph drawn as shown



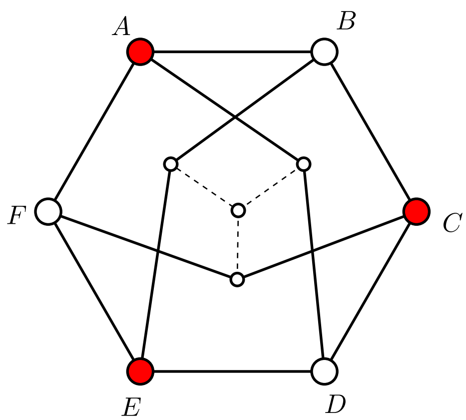
(a) Use Kuratowski's theorem to show that P is nonplanar, that is, show that P contains a subdivision of K_5 or a subdivision of $K_{3,3}$.

(b) Find a spanning tree for P.

(c) The spanning tree you found in part (b) is a planar subgraph of P, how many edges of P have to be inserted into this tree before the resulting graph is nonplanar?

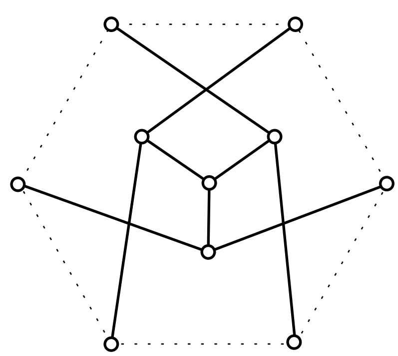
SOLUTION:

(a)



The subdivision of $K_{3,3}$ has A, C and E on one side, with B, D and F on the other.

(b)



(Many other solutions are possible.)

(c) Note that the Petersen graph P is regular of degree 3, and if we remove any 4 edges from the Petersen graph, the resulting graph has at most 5 vertices of degree 3, and therefore cannot contain a subgraph which is a subdivision of $K_{3,3}$. Clearly, since each vertex in K_5 has degree 4, it also cannot contain a subdivision of K_5. Thus, if we remove any 4 edges from the Petersen graph, the resulting graph is planar.

On the other hand, if we remove the three edges incident with the center vertex, the resulting graph contains a subdivision of $K_{3,3}$ as shown in the figure, so that if we remove only 3 edges, there is no guarantee that the resulting graph will be planar.

Therefore, since any spanning tree for the Petersen graph P contains exactly $9 = 10 - 1$ edges, we can add back at most 2 edges and guarantee that the resulting graph is planar.

6. An old puzzle called **The Tower of Hanoi** consists of three pegs, A, B, and C. On the peg A there are n disks of different diameters arranged by decreasing size from the bottom to the top. You wish to transfer all of the n disks from peg A to peg B using the smallest number of moves possible. The rules for moving the disks are as follows:

   Only one disk may be moved at a time, and it may be moved from one peg to either of the other two pegs.

   No disk may be placed on top of one of smaller diameter.

   (a) Explain recursively how to accomplish this, that is, assuming you can accomplish the task for k disks, explain how to do it for $k+1$ disks. Of course, include an initial or base case.
   Hint:

   (b) Let a_n denote the smallest number of moves needed to move n disks from peg A to peg B. Write the recurrence relation for a_n, that is, write down a_1, and express a_{n+1} in terms of a_n.

   (c) Solve the recurrence any way you can.

medskip SOLUTION:

   (a) Taking a clue from the figure, the task can be accomplished recursively by first moving the top $n-1$ disks from peg A to peg C, then moving the n^{th} disk to peg B, and finally moving the top $n-1$ disks from peg C to peg B.

   (b) Clearly, we have

$$a_{n+1} = 2\,a_n + 1, \quad \text{for} \quad n \geq 1$$
$$a_1 = 1.$$

   (c) Adding 1 to both sides of the recurrence relation we have

$$a_{n+1} + 1 = 2(a_n + 1)$$

for $n \geq 1$, and letting $b_n = a_n + 1$ for $n \geq 1$, we have

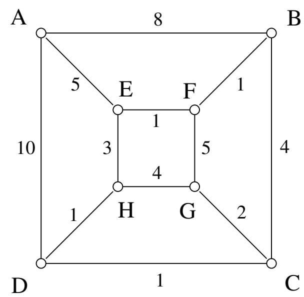
$$b_{n+1} = 2\,b_n, \quad \text{for} \quad n \geq 1$$
$$b_1 = 2.$$

Therefore, an easy induction proof shows that $b_n = 2^n$ for all $n \geq 1$, and the solution is

$$a_n = 2^n - 1$$

for $n \geq 1$.

7. You are given the weighted graph shown.



(a) Use Dijkstra's algorithm to construct a shortest-path spanning tree from the vertex A in the graph on the right. You may do it graphically or by using a table.

(b) For the same weighted graph as in part (a), use Kruskal's algorithm to find a minimal spanning tree. You may do it graphically or by using a table.

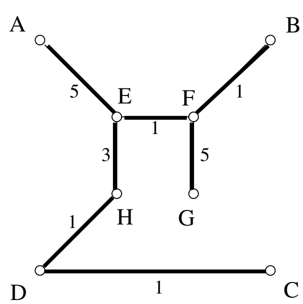
(c) For each of the spanning trees found above, compare the weights of the paths from vertex A to vertex G. Also, compare the total weights of the two spanning trees. What can you conclude?

SOLUTION:

(a) Using Dijkstra's algorithm, starting at A, we find the shortest-path spanning tree for the graph as follows:

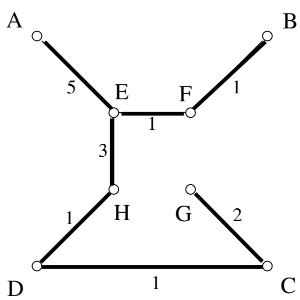
Step	Tree So Far	Fringe Vertices	Candidate Edge (Distance)
1.	A	B, D, E	$AB(8), AD(10), AE(5)$
2.	A, E	B, D, F, H	$AB(8), AD(10), EF(6), EH(8)$
3.	A, E, F	B, D, G, H	$BF(7), AD(10), FG(11), EH(8)$
4.	A, E, F, B	C, D, G, H	$BC(11), AD(10), FG(11), EH(8)$
5.	A, E, F, B, H	C, D, G	$BC(11), DH(9), FG(11)$
6.	A, E, F, B, H, D	C, G	$CD(10), FG(11)$
7.	A, E, F, B, H, D, C	G	$FG(11)$
8.	A, E, F, B, H, D, C, G		

The shortest-path spanning tree from the vertex A, as well as the table, is shown below.



	A	B	C	D	E	F	G	H	
	−, 0	− , ∞	− , ∞	− , ∞	− , ∞	− , ∞	− , ∞	− , ∞	
A		A,8	− , ∞	A,10	A,5		− , ∞	− , ∞	AE
E		A, 8	− , ∞	A,10		E,6	− , ∞	E,8	EF
F		F,7	− , ∞	A,10			F,11	E,8	FB
B			B,12	A,10			F,11	E,8	EH
H			B,12	H,9			F,11		HD
D			D,10				F,11		CD
C							F,11		FG
G									

(b) Using Kruskal's algorithm, the minimal spanning tree for the graph, as well as the table, is shown below.



	A	B	C	D	E	F	G	H
	1	2	3	4	5	6	7	8
BF	1	2	3	4	5	2	7	8
EF	1	2	3	4	2	2	7	8
DC	1	2	3	3	2	2	7	8
DH	1	2	3	3	2	2	7	3
CG	1	2	3	3	2	2	3	3
EH	1	2	2	2	2	2	2	2
BC								
GH								
AE	1	1	1	1	1	1	1	1
FG								
AB								
AD								

(c) As is easily seen from the spanning tree, using Dijkstra'a algorithm we have

$$d(A,G) = 11 \qquad \text{and} \qquad \text{Total Weight} = 17,$$

while using Kruskal's algorithm we have

$$d(A,G) = 12 \qquad \text{and} \qquad \text{Total Weight} = 14.$$

We conclude that the shortest-path spanning tree generated by Dijkstra's algorithm does not have to be a minimal spanning tree, and that the minimal spanning tree generated by Kruskal's algorithm does not have to be a shortest-path spanning tree.