math 22

Assignment 5, due April 13, 2007

Problem 1. A mouse eats its way through a $3 \times 3 \times 3$ cube of cheese by eating all of the $1 \times 1 \times 1$ subcubes. If it starts at a corner subcube and always moves on to an adjacent subcube (sharing a face of area 1), can it do this and eat the center subcube last? Give a method for doing this or prove it is impossible. (Ignore gravity.)

Problem 2. Give a combinatorial argument to find a closed form expression for the following sequence

$$a_n = 2 \cdot 1 \cdot \binom{n}{2} + 3 \cdot 2 \cdot \binom{n}{3} + \cdots + n \cdot (n-1) \cdot \binom{n}{n}$$

for $n = 2, 3, 4, \ldots$.

Hint: Given n people, count the number of ways to form an Ed Leonard fan club which contains a president and a vice-president.

Problem 3. A town jail contains four holding cells. On a particularly busy night, twelve people are arrested. Certain prisoners do not get along with certain others and must be put into separate cells, as shown in the following table.

Prisoner	doesn't get along with
1	3, 5, 8, 9, 10, 11
2	3, 4, 6, 7, 9, 11
3	1, 2, 6, 8, 11, 12
4	2, 5, 6, 8, 10,12
5	1, 4, 7, 9, 10
6	2, 3, 4, 7, 9, 11, 12
7	2, 5, 6, 8, 10
8	1, 3, 4, 7, 12
9	1, 2, 5, 6, 11
10	1, 4, 5, 7, 12
11	1, 2, 3, 6, 9
12	3, 4, 6, 8, 10

(a) Find a way of putting the prisoners into the four cells in such a way as to avoid possible conflicts during the night.

(b) Draw the graph G whose vertices correspond to the prisoners, with an edge between two vertices if and only if the two prisoners share a cell for the night as in part (a).

Problem 4. Let G be a graph with p vertices, show that G has $2^p - 1$ induced subgraphs.

Problem 5. Let $G = (V, E)$ be a graph with vertex set V and edge set E, and let $p = |V|$ be the number of vertices in G, and $q = |E|$ the number of edges in G. The **average degree** of the vertices in G is defined to be

$$A(G) = \frac{1}{p} \sum_{v \in V} \deg(v).$$

If G is a connected graph, what can you say about G if

(a) $A(G) > 2$? (b) $A(G) = 2$? (c) $A(G) < 2$?

Draw a few pictures before committing yourself!!!

Problem 6. Given a graph G, show that the following are equivalent

(a) G is a tree.

(b) G is connected, and the removal of any edge disconnects G.

(c) G has no cycles, and the addition of any new edge creates exactly one cycle.

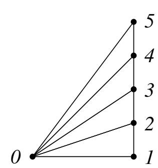
Problem 7. Let $G = (V, E)$ be a connected graph. The graph G is said to be **bipartite** if and only if there is a partition of the vertex set $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$, such that for every edge $ab \in E$, the end vertices are in different sets, that is, either $a \in V_1$ and $b \in V_2$, or $a \in V_2$ and $b \in V_1$.

(a) Show that G is two colorable if and only if it is bipartite.

(b) Show that G is two colorable if and only if it contains no cycles of odd length.

Problem 8. A *fan* of order n is a graph on $n+1$ vertices, labeled $\{0, 1, 2 \ldots, n\}$, with $2n - 1$ edges defined as follows: Vertex 0 is connected by an edge to each of the other n vertices, and vertex k is connected by an edge to vertex $k+1$, for $1 \le k < n$.

For example, the fan of order 5, which has six vertices and nine edges, is shown below.



Let a_n be the number of spanning trees for a fan of order n.

(a) Calculate a_1, a_2, and a_3, and show all the spanning trees in each case.

(b) By observing how the topmost vertex (vertex n) is connected to the rest of the spanning tree, show that a_n satisfies the full-history recurrence relation

$$a_n = 1 + a_{n-1} + \sum_{k=1}^{n-1} a_k$$

for $n \ge 1$, where $a_0 = 0$ and $a_1 = 1$.

(c) Conjecture a value for a_n, for $n \ge 1$, and prove your conjecture is true.