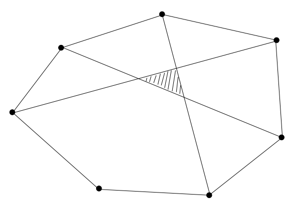
math 222

Assignment 4, due Friday March 23, 2007

Problem 1. How many triangles are formed using chords and sides of a convex $n-$gon, where the vertices of the triangle need not be vertices of the $n-$gon?

For example, one such triangle is shown below.



Assume that no three chords meet at the same interior point.

Hint: Relate different sorts of triangles to different sized sets of vertices of the polygon.

Problem 2. The *Fibonacci Sequence:*

$$0,\ 1,\ 1,\ 2,\ 3,\ 5,\ 8,\ 13,\ 21,\ 34,\ 55,\ 89,\ 144,\ \cdots$$

where each term is the sum of the two preceding terms, satisfies the recurrence relation

$$\begin{aligned} F_{n+2} &= F_{n+1} + F_n \\ F_0 &= 0 \\ F_1 &= 1 \end{aligned}$$

for $n = 0,\ 1,\ 2,\ \ldots.$

(a) Show that

$$F_{n+m+1} = F_{m+1}F_{n+1} + F_m F_n$$

for all $m, n \geq 0$.

(b) Show that

$$F_{3n} = F_{n+1}^3 + F_n^3 - F_{n-1}^3$$

for all $n \geq 1$.

(c) Show that

$$F_{2n} = \binom{n}{0}F_0 + \binom{n}{1}F_1 + \binom{n}{2}F_2 + \binom{n}{3}F_3 + \cdots + \binom{n}{n}F_n$$

for all $n \geq 0$.

Problem 3. There are $2n$ people standing in line at a box office. Admission is one dollar and n of the people have exactly this amount. The other n each have exactly one two dollar coin. Unfortunately, the box office starts off with no change. A sequence of these $2n$ people is said to be *workable* if, up to each point, the number of people with one dollar is not less than the number of people with 2 dollars. In such situations correct change can be given to each person who needs it. How many workable situations are there?

Hint: Let L stand for anyone with a dollar and T stand for anyone with two dollars. The total number of permutations of n T's and n L's is

$$\binom{2n}{n}$$

since each arrangement is determined by which of the $2n$ possible locations are chosen for the n T's. Now count the number of nonworkable permutations and subtract this from the total number to get the number of workable permutations.

Problem 4. For $n \geq 0$, let a_n be the number of regions on the surface of a sphere formed by n great circles, no three of which are concurrent.

(a) What are a_0, a_1, a_2, and a_3 ?

(b) Find a recurrence relation and initial condition satisfied by a_n.

(c) Solve the recurrence relation you found in part (b).

Problem 5. Consider a party attended by n married couples. Suppose that no person shakes hands with his or her spouse, and the $2n-1$ people other than the host shake hands with different numbers of people. With how many people does the hostess shake hands?

Problem 6. Let G be a graph whose vertices correspond to the bit-strings of length n, $a = a_1a_2\cdots a_n$ where $a_i = 0$ or 1, and whose edges are formed by joining those bit-strings which differ in exactly two places.

(a) Show that G is regular, that is, every vertex has the same degree, and find the degree of each vertex.

(b) Find a necessary and sufficient condition that there exist a path joining two vertices $a = a_1a_2\cdots a_n$ and $b = b_1b_2\cdots b_n$ in G.

(c) Find the number of connected components of G.

Problem 7. The queen and her prime minister each live in a complex of underground rooms.
The queen's rooms are 15 in number, 1 for her and 14 for her servants, and they are connected by tunnels. There is at most one tunnel between any two rooms.
For each of the servant's rooms, there is one and only one path that leads to the queen's room.
The prime minister and his cabinet occupy 7 rooms, none of which are the queen's rooms. There is at most one tunnel between any two of the prime minister's rooms.
Together, the underground complexes have a total of 36 tunnels.
Explain why the entire complex is connected.

Problem 8. By using various combinations of the red, green, and blue filters on a spotlight the lighting technician at a theatre can obtain 8 lighting effects. The filters may be changed one at a time by either adding one or removing one. Starting and ending with no filters, how can the technician test all the effects without repeating any effect except the final one.