math 222

Assignment 3, due Friday March 9, 2007

Problem 1.

(a) Find a closed form expression for
$$1 + 2 + 3 + \cdots + n.$$

(b) Make a conjecture about the terms of the following sequence, and prove your conjecture.
$$\frac{1}{1+2}, \qquad \frac{1+2}{2+3+4}, \qquad \frac{1+2+3}{3+4+5+6}, \qquad \frac{1+2+3+4}{4+5+6+7+8}, \qquad \ldots$$

Problem 2. Find a closed form expression for
$$a_n = 1^5 + 2^5 + \cdots + n^5$$
for $n \geq 1$.

Problem 3. For each of the following

(a) $\sum_{k=1}^{n} (-1)^{k-1} k$

(b) $\sum_{k=1}^{n} (-1)^{k-1} k^2$

(c) $\sum_{k=1}^{n} (-1)^{k-1} k(k-1)$

find a closed form expression valid for $n \geq 1$. Justify your answers, using mathematical induction or otherwise.

Problem 4. For each $n \geq 1$, let a_n be the number of ways to group $2n$ people into pairs.

(a) Find a recurrence relation and an initial condition satisfied by the sequence $\{a_n\}_{n\geq 1}$.

(b) Conjecture a value for a_n, and prove your conjecture is true.

Problem 5. Show that
$$a_n = \frac{1}{n+1}\binom{2n}{n}$$
is an integer for $n = 1, 2, 3 \ldots.$

Problem 6. A certain computer system considers a string of bits a valid codeword if and only if it contains an even number of $1's$. For example 1 0 0 0 0 1 is a valid codeword, but 1 0 0 1 0 0 1 is not. Let a_n be the number of valid n-bit codewords.

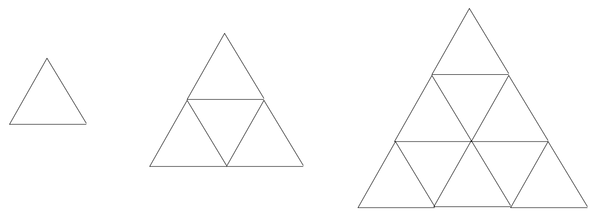
(a) Find a recurrence relation and an initial condition satisfied by a_n.

(b) Given a positive integer N, how many valid codewords of length at most N are there?

Problem 7. A certain basketball team can only sink foul shots and lay-ups, worth 1 and 2 points, respectively. Let a_n denote the number of ways the team can score n points. (Scoring 1 then 2 is considered to be different than scoring 2 then 1). Write down a recurrence relation for a_n with initial conditions for a_0 and a_1, and explain why it holds for all $n \geq 2$. What is the solution to this recurrence relation?

Problem 8. Given a positive integer n, consider an equilateral triangle of side n made up of n^2 equilateral triangles of side 1. Let a_n be the total number of equilateral triangles present for $n \geq 1$.

For example, in figure below,



we have

$$a_1 = 1 \qquad a_2 = 4 + 1 = 5 \qquad a_3 = 9 + 3 + 1 = 13$$

(a) Find a recurrence relation and initial condition satisfied by the sequence $\{a_n\}_{n\geq 1}$.

(b) Conjecture a value for a_n, and prove your conjecture is true.