MATH 222

June 10, 2002

Redei's Theorem and the Camion-Moon Theorem

THEOREM (REDEI). Every tournament has a directed Hamiltonan path.

PROOF. Suppose we have a directed path

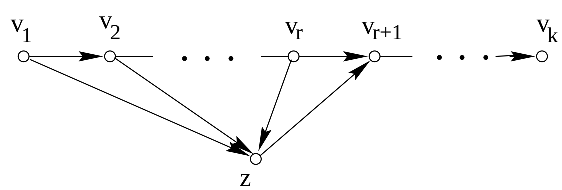
$$v_1,\ v_2,\ \ldots\ v_k$$

which does not contain all the vertices of the graph. Let z be any vertex not on this directed path.

If (z, v_1) is an arc, we can insert z at the beginning. If (z, v_1) is not an arc, since we have a tournament, then (v_1, z) is an arc.

In this case, if (z, v_2) is an arc, then we can insert z between v_1 and v_2 to get a directed path from v_1 to v_k which includes z.

If (z, v_2) is not an arc, since we have a tournament, then (v_2, z) is an arc; and we let r be the greatest integer for which (v_1, z), (v_2, z), $\ldots$ (v_r, z) are arcs.



If $r < k$, we have arcs (v_r, z) and (z, v_{r+1}), so we can insert z between v_r and v_{r+1}. If $r = k$, we can insert z at the end.

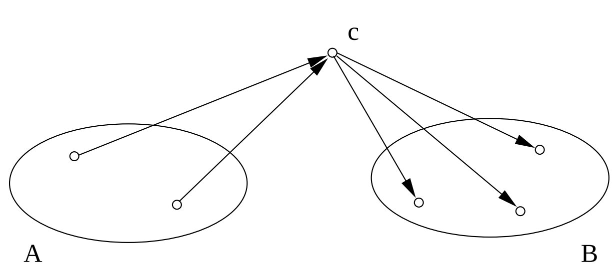
Therefore, if we have a directed path which does not contain all the vertices, we can always insert another vertex into this directed path. Since the graph is finite, we will eventually get a directed path that contains all of the vertices. □

Exercise: Show that every strongly connected directed graph contains a directed cycle.

THEOREM (CAMION-MOON). Every strongly connected tournament has a directed Hamiltonian cycle.

PROOF. Let G be a strongly connected tournament with V vertices. Since G is strongly connected and we have no multiple edges in the underlying graph, then $V \geq 3$. We will prove by mathematical induction that for each k with $3 \leq k \leq V$, the digraph G has a directed cycle of length k.

Base Case: Let c be an arbitrary vertex of G, we divide the remaining vertices into two sets A and B. A vertex a is in A if and only if there is an arc (a, c) from a to c but no arc (c, a) from c to a. Similarly, a vertex b is in B if and only if there is an arc (c, b) from c to b but no arc (b, c) from b to c.

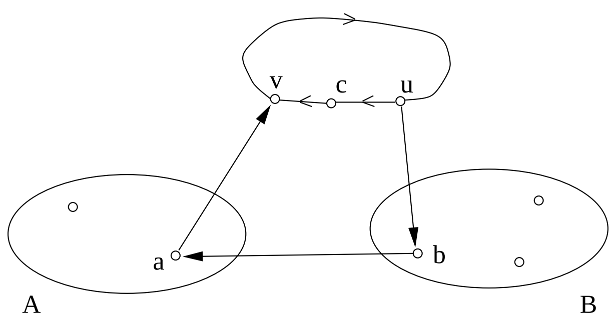


Now, $V \geq 3$, so that both A and B must be nonempty. If $B = \emptyset$, for example, then there would be no directed path from c to any vertex in A, which contradicts the fact that G is strongly connected.

Also, if all arcs between vertices in A and vertices in B go from a vertex in A to a vertex in B, then again there would be no directed path from any vertex in B to any vertex in A, again contradicting the fact that G is strongly connected. Therefore, there is an arc from some vertex $b \in B$ to a vertex $a \in A$, and we have a directed cycle of length 3 given by $a \to c \to b \to a$.

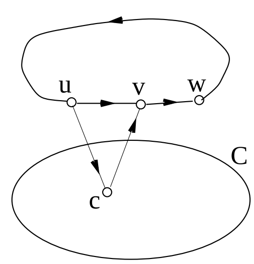
Inductive Step: Now suppose that the digraph G has a directed cycle of length k for some k with $3 \leq k < V$, we will show that this implies that G has a directed cycle of length $k+1$. We divide the $V - k$ vertices which are not in the cycle into three sets A, B, and C, where vertices in A have no arcs coming from the cycle, vertices in B have no arcs going to the cycle, and vertices in C have arcs coming from the cycle and arcs going to the cycle. We consider two cases.

1. Suppose C is empty, then both A and B must be nonempty, otherwise G would not be strongly connected. As in the base case, there is an arc going from some vertex $b \in B$ to some vertex $a \in A$.



   If c is any vertex in the cycle, we let u and v be the vertices immmediately preceding and immediately following c on the cycle, respectively. If we replace the portion of the cycle $u \to c \to v$ by the directed path $u \to b \to a \to v$, we obtain a directed cycle of length $k+1$ as desired.

2. Suppose that C is not empty, and let $c \in C$. Let u be a vertex on the cycle such that (u, c) is an arc, and let v be the vertex immediately following u on the cycle. If (c, v) is an arc, then we can insert c into the cycle between u and v.



   If (c, v) is not an arc, then since we have a tournament, (v, c) is an arc, and we look at the vertex w immmediately following v on the cycle, if (c, w) is an arc, then we can insert c into the cycle between v and w.

   Proceeding as in the proof of Redei's theorem, we will eventually find a place to insert c into the cycle between consecutive vertices in the cycle. Thus, in this case also, we obtain a directed cycle of length $k+1$ as desired.

From the principle of mathematical induction for each $3 \leq k \leq V$, the directed graph G has a directed cycle of length k. $\square$