Dirichlet's Pigeonhole Principle

The pigeonhole principle states that if $n+1$ pigeons occupy n pigeonholes, then at least one pigeonhole will contain at least two pigeons. It may be that all $n+1$ pigeons are in the same pigeonhole, or every pigeonhole might have exactly three pigeons, but in any case, at least one pigeonhole will contain more than one pigeon.

Example 1. Given a rational number a/b where a and b are both assumed to be positive, then the decimal expansion of a/b either terminates or repeats.
A terminating decimal is one of the form 1.2345, and a repeating decimal is one of the form $1.23454545\cdots$ and is sometimes written as $1.23\overline{45}$.
SOLUTION: Let $r_0 = a$ and let

$$r_1,\ r_2,\ r_3,\ \ldots$$

be the successive remainders from long division when a is divided by b. From the division algorithm, each of these remainders satisfies

$$0 \le r_k \le b-1.$$

If one of the remainders $r_i = 0$, then the division terminates and the fraction a/b has a terminating decimal expansion.
If none of the remainders is zero, then the sequence of remainders continues forever, and by the Pigeonhole Principle, some remainder must repeat, that is, $r_j = r_k$ for some integers j and k with $j < k$. Thus, the decimal digits obtained from the divisions between r_j and r_{k-1} repeat forever, and the fraction a/b has a repeating decimal expansion. □

We can extend the pigeonhole principle by noting that if $2n+1$ pigeons fit into n pigeonholes, then at least one pigeonhole contains more than two pigeons. Also, if there are $3n+1$ pigeons in n pigeonholes, then at least one pigeonhole contains more than three pigeons. In general, we have the following.

Theorem. If m pigeons occupy n pigeonholes, then at least one pigeonhole contains

$$\left\lfloor \frac{m-1}{n} \right\rfloor + 1$$

pigeons. Here $\left\lfloor \frac{m-1}{n} \right\rfloor$ is the greatest integer less than or equal to $(m-1)/n$.

proof. The largest multiple of n less than m is found by dividing $m-1$ by n and discarding the fractional part, that is,

$$\left\lfloor \frac{m-1}{n} \right\rfloor.$$

If we had exactly $n\left\lfloor \frac{m-1}{n} \right\rfloor$ pigeons, since

$$n\left\lfloor \frac{m-1}{n} \right\rfloor \le m-1 < m,$$

and we have m pigeons, we could put $\left\lfloor \frac{m-1}{n} \right\rfloor$ in each pigeonhole and have some left over. Therefore, at least one pigeonhole contains more than this number of pigeons.

□

Example 2. Now we use the generalized pigeonhole principle to show the following:

(a) Given a sequence $a_1,\ a_2,\ a_3,\ \ldots\ a_{n^2+1}$ of any n^2+1 different positive integers, either there is an increasing subsequence of $n+1$ terms, or else there is a decreasing sequence of $n+1$ terms.

For example, if $n = 3$, then any sequence of 10 different positive integers either contains an increasing sequence of four terms or else a decreasing sequence of four terms.

(b) The result in part (a) is the best possible in the sense that it is not true for any shorter original sequence, that is, if we start with only n^2 different positive integers, it is possible to have no increasing or decreasing sequence of $n+1$ terms.

SOLUTION:

(a) Let the sequence be $a_1,\ a_2,\ a_3,\ \ldots\ a_{n^2+1}$, and associate with each term a_k the positive integer t_k which gives the length of the longest increasing subsequence which starts at a_k. Thus, there are n^2+1 integers t_k.

Now, if any of the t_k's are $n+1$ or larger, then we have found an increasing subsequence of length at least $n+1$.

On the other hand, if all of the t_k's are less than $n+1$, then each t_k has a value between 1 and n. Therefore, we have n^2+1 pigeons (the t_k's) which we want to put into n pigeonholes (the values of the t_k's).

From Dirichlet's pigeonhole principle, one of these pigeonholes must contain at least

$$\left\lfloor \frac{(n^2+1)-1}{n} \right\rfloor + 1 = n+1$$

pigeons, that is, at least $n+1$ of the t_k's must be equal. Now we will show that the a_k's associated with these equal t_k's must form a decreasing subsequence.

If $i < j$ and a_i and a_j have equal t's, then we must have $a_i > a_j$. Otherwise, if $a_i \leq a_j$, and we append a_i onto the front of the increasing subsequence starting at a_j, then we form an increasing subsequence starting at a_i which has length $t_j+1 = t_i+1 > t_i$. However, this contradicts the definition of t_i, therefore the a_k's associated with the equal t_k's form a decreasing subsequence of length $n+1$.

(b) In order to show that this is the best result possible, we give a sequence of n^2 terms which has no increasing subsequence of length $n+1$ and which also has no decreasing subsequence of length $n+1$. The sequence is as follows:

$$n, n-1, \ldots, 1, 2n, 2n-1, \ldots, n+1, 3n, 3n-1, \ldots, 2n+1, \ldots, n^2, n^2-1, \ldots, (n-1)n+1.$$

Here the numbers from 1 to n^2 are arranged in a pattern such that the longest increasing subsequences are all n terms long, and the longest decreasing subsequences are also all n terms long.

Note that if we try to insert the number n^2+1 anywhere into this sequence, we will create either an increasing subsequence of length $n+1$ or a decreasing subsequence of length $n+1$. □