MATH 214 (R1) Winter 2008
Intermediate Calculus I

Solutions to Problem Set #9

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Question 1. [Sec. 14.2, # 8] Given the vector equation

$$\mathbf{r}(t) = 2\sin t\,\mathbf{i} + 3\cos t\,\mathbf{j},$$

(a) sketch the plane curve with the given vector equation,

(b) find $\mathbf{r}'(t)$,

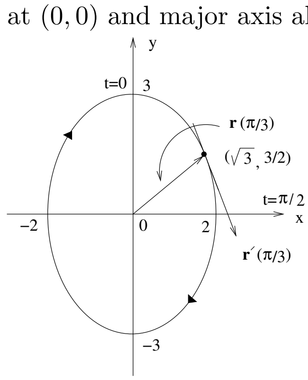
(c) sketch the position vector $\mathbf{r}(t)$ and the tangent vector $\mathbf{r}'(t)$ for the value $t = \pi/3$.

Solution:

(a) $x = 2\sin t$, $y = 3\cos t$ implies $\sin t = x/2$, and $\cos t = y/3$, so that

$$\sin^2 t + \cos^2 t = \frac{x^2}{4} + \frac{y^2}{9} = 1.$$

The curve is an ellipse with center at $(0, 0)$ and major axis along the y-axis.



(b) $\mathbf{r}'(t) = \langle 2\cos t, -3\sin t\rangle$.

(c) See the graph above.

Question 2. [Sec. 14.2, # 26] Find parametric equations for the tangent line to the curve whose parametric equations are

$$x = \ln t, \quad y = 2\sqrt{t}, \quad z = t^2, \quad 0 < t < \infty$$

at the point $(0, 2, 1)$.

Solutions: We have $\mathbf{r}(t) = \langle \ln t, 2\sqrt{t}, t^2\rangle$, so that $\mathbf{r}'(t) = \langle 1/t, 1/\sqrt{t}, 2t\rangle$.

At $(0, 2, 1)$, $t = 1$, so that $\mathbf{r}'(1) = \langle 1, 1, 2\rangle$ is a direction vector for the tangent line whose parametric equations are

$$x = t, \quad y = 2 + t, \quad z = 1 + 2t.$$

Question 3. [Sec. 14.2, # 30] Given the curve

$$\mathbf{r}(t) = \langle \sin \pi t, 2 \sin \pi t, \cos \pi t \rangle ,$$

(a) Find the point of intersection of the tangent lines to the curve at the points where $t = 0$ and $t = 0.5$.

(b) Illustrate by graphing the curve and both tangent lines.

SOLUTION:

(a) We first find $\mathbf{r}'(t) = \langle \pi \cos \pi t, 2\pi \cos \pi t, -\pi \sin \pi t \rangle$. We can use this for direction vectors for the 2 tangent lines.

Let $t = 0$. $\mathbf{r}'(0) = \langle \pi, 2\pi, 0 \rangle$. The point on the curve is $(0, 0, 1)$, and the tangent line is

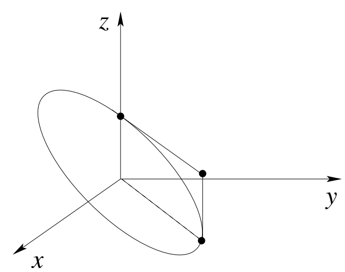
$$x = \pi t, \quad y = 2\pi t, \quad z = 1.$$

Let $t = 0.5$. $\mathbf{r}'(1/2) = \langle 0, 0, -\pi \rangle$. The point on the curve is $(1, 2, 0)$, and the tangent line is

$$x = 1, \quad y = 2, \quad z = -\pi s.$$

At the point of intersection of these tangent lines: $x: \ \pi t = 1$ implies $t = 1/\pi$ and $z: \ -\pi s = 1$ implies $s = -1/\pi$, so that the point of intersection is $(1, 2, 1)$.

(b) The graph and the two tangent lines are sketched below.



Question 4. [Sec. 14.2, # 40] Find $\mathbf{r}(t)$ if

$$\mathbf{r}'(t) = \sin t \, \mathbf{i} - \cos t \, \mathbf{j} + 2t \, \mathbf{k} \qquad \text{and} \qquad \mathbf{r}(0) = \mathbf{i} + \mathbf{j} + 2\, \mathbf{k}.$$

SOLUTION: We have

$$\mathbf{r}(t) = \int \mathbf{r}'(t) \, dt = -\cos t \, \mathbf{i} - \sin t \, \mathbf{j} + t^2 \, \mathbf{k} + \mathbf{C} = \langle -\cos t, -\sin t, t^2 \rangle + \langle a, b, c \rangle,$$

and since $\mathbf{r}(0) = \langle 1, 1, 2 \rangle$ at $t = 0$, we have

$$\langle 1, 1, 2 \rangle = \langle -\cos 0, -\sin 0, 0^2 \rangle + \langle a, b, c \rangle = \langle -1, 0, 0 \rangle + \langle a, b, c \rangle$$

therefore

$$\langle a, b, c \rangle = \langle 2, 1, 2 \rangle$$

therefore

$$\mathbf{r}(t) = \langle -\cos t, -\sin t, t^2 \rangle + \langle 2, 1, 2 \rangle = (2 - \cos t) \, \mathbf{i} + (1 - \sin t) \, \mathbf{j} + (2 + t^2) \, \mathbf{k}.$$

Question 5. **[Sec. 14.3, # 2]** Find the length of the curve

$$\mathbf{r}(t) = \left\langle t^2, \sin t - t\cos t, cost + t\sin t \right\rangle, \quad 0 \le t \le \pi.$$

SOLUTION: We have

$$\mathbf{r}'(t) = \langle 2t, \cos t - \cos t + t\sin t, -\sin t + \sin t + t\cos t\rangle = \langle 2t, t\sin t, t\cos t\rangle$$

so that

$$|\mathbf{r}'(t)| = \sqrt{4t^2 + t^2\sin^2 t + t^2\cos^2 t} = \sqrt{4t^2 + t^2(\sin^2 t + \cos^2 t)} = \sqrt{5t^2} = \sqrt{5}\,t$$

since $t \in [0, \pi]$. Therefore, the length of the curve is

$$L = \int_0^\pi |\mathbf{r}'(t)|\,dt = \int_0^\pi \sqrt{5}\,t\,dt = \sqrt{5}\,\frac{t^2}{2}\bigg|_0^\pi = \frac{\sqrt{5}}{2}\pi^2.$$

Question 6. **[Sec. 14.3, # 14]** Given the curve

$$\mathbf{r}(t) = \left\langle t^2, \sin t - t\cos t, cost + t\sin t \right\rangle, \quad t > 0$$

(a) Find the unit tangent and unit normal vectors $\mathbf{T}(t)$ and $\mathbf{N}(t)$.

(b) Use the formula

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|}$$

to find the curvature.

SOLUTION:

(a) From the previous problem, $\mathbf{r}'(t) = \langle 2t, t\sin t, t\cos t\rangle$ and $|\mathbf{r}'(t)| = \sqrt{5}\,t$, so that

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{\langle 2t, t\sin t, t\cos t\rangle}{\sqrt{5}t} = \left\langle \frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\sin t, \frac{1}{\sqrt{5}}\cos t \right\rangle,$$

and so

$$\mathbf{T}'(t) = \left\langle 0, \frac{1}{\sqrt{5}}\cos t, -\frac{1}{\sqrt{5}}\sin t \right\rangle$$

which implies

$$|\mathbf{T}'(t)| = \sqrt{\frac{1}{5}(\cos^2 t + \sin^2 t)} = \frac{1}{\sqrt{5}},$$

and therefore

$$\mathbf{N}(t) = \frac{\mathbf{T}'(t)}{|\mathbf{T}'(t)|} = \frac{\langle 0, \frac{1}{\sqrt{5}}\cos t, -\frac{1}{\sqrt{5}}\sin t\rangle}{1/\sqrt{5}} = \langle 0, \cos t, -\sin t\rangle.$$

(b) The curvature is

$$\kappa(t) = \frac{|\mathbf{T}'(t)|}{|\mathbf{r}'(t)|} = \frac{1/\sqrt{5}}{\sqrt{5}\,t} = \frac{1}{5t}.$$

Question 7. [Sec. 14.3, # 18] Given the curve

$$\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + (1+t^2)\,\mathbf{k}$$

use the formula

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3}$$

to find the curvature.

SOLUTION: We have

$$\mathbf{r}(t) = \langle t, t, 1+t^2\rangle,$$

and

$$\mathbf{r}'(t) = \langle 1, 1, 2t\rangle,$$

and

$$\mathbf{r}''(t) = \langle 0, 0, 2\rangle.$$

Therefore

$$\mathbf{r}'(t)\times\mathbf{r}''(t) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 2t \\ 0 & 0 & 2 \end{vmatrix} = 2\mathbf{i} - 2\mathbf{j} = \langle 2, -2, 0\rangle$$

and so

$$|\mathbf{r}'\times\mathbf{r}''| = \sqrt{4+4} = 2\sqrt{2},$$

and since

$$|\mathbf{r}'| = \sqrt{1+1+4t^2} = \sqrt{2+4t^2} = \sqrt{2}\sqrt{1+2t^2}$$

then

$$\kappa(t) = \frac{|\mathbf{r}'(t)\times\mathbf{r}''(t)|}{|\mathbf{r}'(t)|^3} = \frac{2\sqrt{2}}{(\sqrt{2})^3(1+2t^2)^{\frac{3}{2}}} = \frac{1}{(1+2t^2)^{\frac{3}{2}}}.$$

Question 8. [Sec. 14.3, # 26] Given the curve $y = \ln x$, at what point does the curve have maximum curvature? What happens to the curvature as $x\to\infty$?

SOLUTION: We use $\kappa(x) = |f''(x)|/(1+(f'(x))^2)^{3/2}$.

Since $y = \ln x$, then $y' = \dfrac{1}{x}$, which implies that $y'' = -\dfrac{1}{x^2}$ for $x > 0$. Therefore

$$\kappa(x) = \frac{|-\frac{1}{x^2}|}{(1+\frac{1}{x^2})^{\frac{3}{2}}} = \frac{1/x^2}{(1+\frac{1}{x^2})^{\frac{3}{2}}} = \frac{1/x^2}{(x^2+1)^{\frac{3}{2}}/x^3} = \frac{x}{(1+x^2)^{\frac{3}{2}}}$$

and so

$$\kappa'(x) = \frac{(1+x^2)^{\frac{3}{2}} - x\cdot\frac{3}{2}(1+x^2)^{\frac{1}{2}}(2x)}{(1+x^2)^3} = \frac{(1+x^2)^{\frac{1}{2}}[1+x^2-3x^2]}{(1+x^2)^3} = \frac{1-2x^2}{(1+x^2)^{\frac{5}{2}}}.$$

The critical point is $x = 1/\sqrt{2}$ (remember the domain of f is $x > 0$). Then on $(0, 1/\sqrt{2})$, $\kappa'(x) > 0$ so κ is increasing; and on $(1/\sqrt{2}, \infty)$, $\kappa'(x) < 0$ so κ is decreasing. Hence the curvature is a maximum at $x = 1/\sqrt{2}$. The maximum curvature occurs at $(1/\sqrt{2}, \ln(1/\sqrt{2}))$. Also,

$$\lim_{x\to\infty}\kappa(x) = \lim_{x\to\infty}\frac{1-2x^2}{(1+x^2)^{\frac{5}{2}}} = \lim_{x\to\infty}\frac{x^2\left(\frac{1}{x^2}-2\right)}{\left(x^2\left(\frac{1}{x^2}+1\right)\right)^{\frac{5}{2}}}$$

$$= \lim_{x\to\infty}\frac{x^2\left(\frac{1}{x^2}-2\right)}{x^5\left(\frac{1}{x^2}+1\right)^{\frac{5}{2}}} = \lim_{x\to\infty}\frac{\frac{1}{x^2}-2}{x^3\left(\frac{1}{x^2}+1\right)^{\frac{5}{2}}} = \lim_{x\to\infty}\frac{-2}{x^3} = 0.$$

Question 9. [Sec. 14.3, # 42] Find the equations of the normal plane and the osculating plane of the curve

$$x = t, \qquad y = t^2, \qquad z = t^3$$

at the point $(1, 1, 1)$.

Solution: At $(1, 1, 1)$, $t = 1$. We have $\mathbf{r}(t) = \langle t, t^2, t^3 \rangle$ and $\mathbf{r}'(t) = \langle 1, 2t, 3t^2 \rangle$.

The normal plane is determined by the vectors $\mathbf{B}$ and $\mathbf{N}$ so a normal vector is the unit tangent vector $\mathbf{T}$ (or $\mathbf{r}'$.

Now

$$\mathbf{T}(1) = \frac{\mathbf{r}'(1)}{|\mathbf{r}'(1)|} = \frac{\langle 1, 2, 3 \rangle}{\sqrt{1+4+9}} = \frac{1}{\sqrt{14}} \langle 1, 2, 3 \rangle.$$

Using $\langle 1, 2, 3 \rangle$ and the point $(1, 1, 1)$, an equation of the normal plane is

$$x - 1 + 2(y-1) + 3(z-1) = 0$$

which implies $x + 2y + 3z = 6$.

The osculating plane is determined by the vectors $\mathbf{N}$ and $\mathbf{T}$, and we can use for a normal vector

$$\mathbf{n} = \mathbf{B} = \mathbf{T} \times \mathbf{N}.$$

Now

$$\mathbf{T}(t) = \frac{\mathbf{r}'(t)}{|\mathbf{r}'(t)|} = \frac{1}{\sqrt{1+4t^2+9t^4}} \langle 1, 2t, 3t^2 \rangle$$

which implies

$$\mathbf{T}'(t) = \frac{1}{2}(1+4t^2+9t^4)^{-\frac{3}{2}}(8t+36t^3)\langle 1, 2t, 3t^2 \rangle + \frac{1}{\sqrt{1+4t^2+9t^4}} \langle 0, 2, 6t \rangle,$$

and this implies that

$$\mathbf{T}'(1) = \frac{1}{2} \frac{8+36}{(\sqrt{1+4+9})^3} \langle 1, 2, 3 \rangle + \frac{1}{\sqrt{1+4+9}} \langle 0, 2, 6 \rangle = \frac{1}{7\sqrt{14}} \langle 11, 8, -9 \rangle,$$

and therefore

$$\mathbf{N}(1) = \frac{\frac{1}{7\sqrt{14}} \langle 11, 8, -9 \rangle}{\sqrt{121+64+81}} = \frac{\langle 11, 8, -9 \rangle}{\sqrt{266}}$$

For a normal vector use

$$\mathbf{n} = \langle 1, 2, 3 \rangle \times \langle 11, 8, -9 \rangle = \langle -42, 42, -14 \rangle = 14\langle 3, -3, 1 \rangle,$$

then the osculating plane has equation

$$3(x-1) - 3(y-1) + (z-1) = 0,$$

that is, $3x - 3y + z = 1$.

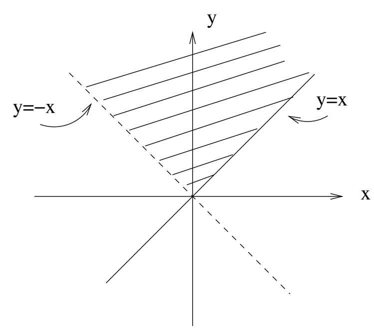
Question 10. [Sec. 15.1, # 16] Find and sketch the domain of the function

$$f(x,y) = \sqrt{y-x}\,\ln(y+x).$$

SOLUTION: The domain of f is

$$D = \{(x,y)\,|\,y \geq x \text{ and } y > -x\} = \{(x,y)\,|\,-y < x \leq y,\ y > 0\}.$$

The graph of D is

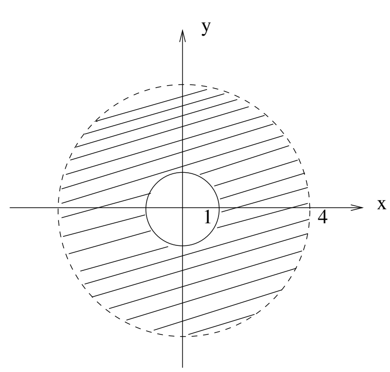


Question 11. [Sec. 15.1, # 18] Find and sketch the domain of the function

$$f(x,y) = \sqrt{x^2+y^2-1} + \ln(4-x^2-y^2).$$

SOLUTION: For the domain of f we need $x^2+y^2-1 \geq 0$, i.e., $x^2+y^2 \geq 1$ and $4-x^2-y^2 > 0$, i.e., $x^2+y^2 < 4$. So

$$D = \{(x,y)\,|\,1 \leq x^2+y^2 < 4\}$$

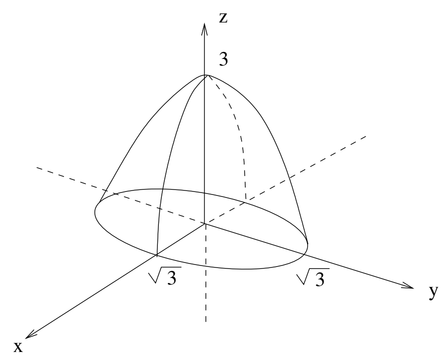


Question 12. [Sec. 15.1, # 26] Sketch the graph of the function

$$f(x, y) = 3 - x^2 - y^2.$$

SOLUTION: Let $z = 3 - x^2 - y^2$. We look at various traces of of f.

$$\begin{aligned}
z = 0: &\quad x^2 + y^2 = 3 \\
z = k: &\quad x^2 + y^2 = 3 - k \quad \text{(a family of circles, } k \le 3) \\
x = 0: &\quad z - 3 = -y^2 \\
x = k: &\quad z - 3 + k^2 = -y^2 \quad \text{(a family of parabolas, opens down)} \\
y = 0: &\quad z - 3 = -x^2 \\
y = k: &\quad z - 3 + k^2 = -x^2 \quad \text{(a family of parabolas, opens down)}
\end{aligned}$$



Question 13. [Sec. 15.1, # 38] Draw a contour map of the function

$$f(x, y) = x^2 - y^2$$

showing several level curves.

SOLUTION: The level curves are $x^2 - y^2 = k$ s.t.

$$\begin{aligned}
k = 0: &\quad x^2 - y^2 = 0 \implies y^2 = x^2 \implies y = \pm x \\
k > 0: &\quad \frac{x^2}{k} - \frac{y^2}{k} = 1 \quad \text{(a family of hyperbolas, } x\text{-int: } x = \pm\sqrt{k}) \\
k < 0: &\quad \frac{x^2}{k} - \frac{y^2}{k} = 1 \quad \text{(a family of hyperbolas, } y\text{-int: } y = \pm\sqrt{k})
\end{aligned}$$