MATH 214 (R1) Winter 2008
Intermediate Calculus I

Solutions to Problem Set #8

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Question 1. **[Sec. 13.4, # 16]** Find two unit vectors orthogonal to both $\mathbf{i} + \mathbf{j} + \mathbf{k}$ and $2\mathbf{i} + \mathbf{k}$.

SOLUTION: Let $\mathbf{a} = \langle 1, 1, 1\rangle$ and $\mathbf{b} = \langle 2, 0, 1\rangle$, then

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 1 \\ 2 & 0 & 1 \end{vmatrix} = \mathbf{i} + \mathbf{j} - 2\mathbf{k}.$$

Therefore $\langle 1, 1, -2\rangle$ is orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ and $|\langle 1, 1, -2\rangle| = \sqrt{1+1+4} = \sqrt{6}$. One unit vector orthogonal to both $\mathbf{a}$ and $\mathbf{b}$ is $\langle 1/\sqrt{6}, 1/\sqrt{6}, -2/\sqrt{6}\rangle$ and the other (going in the opposite direction) is $\langle -1/\sqrt{6}, -1/\sqrt{6}, 2/\sqrt{6}\rangle$.

Question 2. **[Sec. 13.4, # 24]** Find the area of the parallelogram with vertices $K\,(1, 2, 3)$, $L\,(1, 3, 6)$, $M\,(3, 8, 6)$, and $N\,(3, 7, 3)$.

SOLUTION: The parallelogram is determined by the vectors $\mathbf{a} = \overrightarrow{KL} = \langle 0, 1, 3\rangle$ and $\mathbf{b} = \overrightarrow{KN} = \langle 2, 5, 0\rangle$ since

$$\overrightarrow{KL} = \langle 0, 1, 3\rangle, \quad \overrightarrow{KN} = \langle 2, 5, 0\rangle, \quad \overrightarrow{LM} = \langle 2, 5, 0\rangle, \quad \overrightarrow{NM} = \langle 0, 1, 3\rangle.$$

So $KL \parallel NM$ and $KN \parallel LM$. Then the area $A = |\mathbf{a} \times \mathbf{b}|$,

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{vmatrix} = -15\mathbf{i} + 6\mathbf{j} - 2\mathbf{k},$$

therefore

$$A = |\mathbf{a} \times \mathbf{b}| = \sqrt{(-15)^2 + 6^2 + (-2)^2} = \sqrt{265}.$$

Question 3. **[Sec. 13.4, # 28]** Given the points $P\,(2, 0, -3)$, $Q\,(3, 1, 0)$, and $R\,(5, 2, 2)$,

(a) find a vector orthogonal to the plane through the points P, Q, and R, and

(b) find the area of triangle $\triangle PQR$.

SOLUTION:

(a) We have $\overrightarrow{PQ} = \langle 1, 1, 3\rangle$ and $\overrightarrow{PR} = \langle 3, 2, 5\rangle$, so that

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 3 \\ 3 & 2 & 5 \end{vmatrix} = -\mathbf{i} + 4\mathbf{j} - \mathbf{k} = \langle -1, 4, -1\rangle.$$

(b) We have

$$A = \frac{1}{2}|\overrightarrow{PQ} \times \overrightarrow{PR}| = \frac{1}{2}\sqrt{1+16+1} = \frac{\sqrt{18}}{2} = \frac{3\sqrt{2}}{2}.$$

Question 4. [Sec. 13.4, # 32] Given the points

$$P\,(0,1,2),\quad Q\,(2,4,5),\quad R\,(-1,0,1),\quad S\,(6,-1,4),$$

find the volume of the parallelopiped with adjacent edges PQ, PR, and PS.

SOLUTION: Let $\mathbf{a} = \overrightarrow{PQ} = \langle 2,3,3\rangle$, $\mathbf{b} = \overrightarrow{PR} = \langle -1,-1,-1\rangle$, $\mathbf{c} = \overrightarrow{PS} = \langle 6,-2,2\rangle$, then

$$V = |\mathbf{c}\cdot(\mathbf{a}\times\mathbf{b})| = \begin{vmatrix} 6 & -2 & 2 \\ 2 & 3 & 3 \\ -1 & -1 & -1 \end{vmatrix} = |6(-3+3)+2(-2+3)+2(-2+3)| = |0+2+2| = 4.$$

Question 5. [Sec. 13.5, # 10] Find parametric equations and symmetric equations for the line through the point $(2,1,0)$ and perpendicular to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$.

SOLUTION: Let $\mathbf{a} = \mathbf{i}+\mathbf{j} = \langle 1,1,0\rangle$ and $\mathbf{b} = \mathbf{j}+\mathbf{k} = \langle 0,1,1\rangle$, then the vector $\mathbf{a}\times\mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, so that

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}-\mathbf{j}+\mathbf{k} = \langle 1,-1,1\rangle.$$

This vector is the direction vector of the line, and parametric equations of the line through the point $(2,1,0)$ are

$$x = 2+t,\quad y = 1-t,\quad z = t,$$

while the symmetric equations are

$$x-2 = \frac{y-1}{-1} = z.$$

Question 6. [Sec. 13.5, # 16]

(a) Find parametric equations for the line through $(5,1,0)$ that is perpendicular to the plane $2x-y+z=1$.

(b) In what points does this line intersect the coordinate planes?

SOLUTION:

(a) Take as direction vector $\langle 2,-1,1\rangle$ (a normal vector of the plane), then the equation is

$$x = 5+2t,\quad y = 1-t,\quad z = t.$$

(b) For the xy-plane: $z=0$ implies $t=0$, which implies $x=5$, $y=1$, and the line intersects the xy-plane at $(5,1,0)$.

For the yz-plane: $x=0$ implies $5+2t=0$, which implies $2t=-5$, which implies $t=-5/2$, so $y = 1+(5/2) = 7/2$ and $z=-5/2$. The point of intersection is $(0,7/2,-5/2)$.

For the xz-plane: $y=0$ implies $t=1$, which implies $z=1$ and $x=7$, so the point of intersection is $(7,0,1)$.

Question 7. [Sec. 13.5, # 24] Find an equation of the plane through the point $(4, 0, -3)$ and with normal vector $\mathbf{j} + 2\mathbf{k}$.

SOLUTION: The normal vector is $\mathbf{n} = \langle 0, 1, 2i \rangle$, and a point on the plane is $(4, 0, -3)$. The plane has an equation

$$0(x-4) + (y-0) + 2(z+3) = 0,$$

that is $y + 2z = -6$.

Question 8. [Sec. 13.5, # 34] Find an equation of the plane that passes through the point $(1, 2, 3)$ and contains the line

$$x = 3t, \quad y = 1 + t, \quad z = 2 - t, \quad -\infty < t < \infty.$$

SOLUTION: One vector in the plane is $\mathbf{a} = \langle 3, 1, -1 \rangle$ (a direction vector of the line). The points $(0, 1, 2)$ and $(1, 2, 3)$ are in the plane. Let $\mathbf{b} = \langle 1, 1, 1 \rangle$ (from $(0, 1, 2)$ to $(1, 2, 3)$), then

$$\mathbf{n} = \mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = 2\mathbf{i} - 4\mathbf{j} + 2\mathbf{k} = 2\langle 1, -2, 1 \rangle.$$

Using $\langle 1, -2, 1 \rangle$ and the point $(1, 2, 3)$, we obtain the plane

$$(x-1) - 2(y-2) + (z-3) = 0,$$

that is, $x - 2y + z = 0$.

Question 9. [Sec. 13.5, # 46] Given the planes

$$2z = 4y - x \qquad \text{and} \qquad 3x - 12y + 6x = 1,$$

determine whether the planes are parallel, perpendicular, or neither. If neither, find the angle between them.

SOLUTION: Let normal vectors be $\mathbf{n}_1$ and $\mathbf{n}_2$, then

$$\begin{aligned} x - 4y + 2z = 0 \quad &\text{implies} \quad \mathbf{n}_1 = \langle 1, -4, 2 \rangle, \\ 3x - 12y + 6z = 1 \quad &\text{implies} \quad \mathbf{n}_2 = \langle 3, -12, 6 \rangle = 3\langle 1, -4, 2 \rangle, \end{aligned}$$

and since $\mathbf{n}_1 = (1/3)\mathbf{n}_2$, the planes are parallel.

Question 10. [Sec. 13.5, # 54] Find parametric equations for the line of intersection of the planes

$$2x + 5z + 3 = 0 \qquad \text{and} \qquad x - 3y + z + 2 = 0.$$

SOLUTION: Let $\mathbf{n}_1 = \langle 2, 0, 5 \rangle$ and $\mathbf{n}_2 = \langle 1, -3, 1 \rangle$ be normal vectors. The line of intersection is perpendicular to both of these vectors, so a direction vector for the line is

$$\mathbf{n}_1 \times \mathbf{n}_2 = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 0 & 5 \\ 1 & -3 & 1 \end{vmatrix} = 15\mathbf{i} + 3\mathbf{j} - 6\mathbf{k} = 3\langle 5, 1, -2 \rangle.$$

We will use the vector $\langle 5, 1, -2 \rangle$ for a direction vector. Now we find a point in the line, i.e., an intersection point of the given planes. Choosing $z = 0$ in the plane $2x + 5z = -3$, we get $x = -3/2$, then from the 2nd plane $y = 1/6$, so we use the point $(-3/2, 1/6, 0)$.

The parametric equations of the line through the point $(-3/2, 1/6, 0)$ with direction vector $\overrightarrow{v} = \langle 5, 1, -2 \rangle$ is

$$x = -\frac{3}{2} + 5t, \quad y = \frac{1}{6} + t, \quad z = -2t$$

is one answer (you get different equations by choosing different points).

Question 11. [Sec. 14.1, # 4] Find the limit

$$\lim_{t\to 0}\left\langle \frac{e^t-1}{t}, \frac{\sqrt{1+t}-1}{t}, \frac{3}{1+t}\right\rangle.$$

SOLUTION: We find the limit of each component function.

$$\lim_{t\to 0}\frac{e^t-1}{t} \overset{\text{l'H}}{=} \lim_{t\to 0}\frac{e^t}{1} = 1$$

$$\lim_{t\to 0}\frac{\sqrt{1+t}-1}{t} = \lim_{t\to 0}\frac{\sqrt{1+t}-1}{t}\cdot\frac{\sqrt{1+t}+1}{\sqrt{1+t}+1} = \lim_{t\to 0}\frac{1+t-1}{t(\sqrt{1+t}+1)}$$

$$= \lim_{t\to 0}\frac{1}{\sqrt{1+t}+1} = \frac{1}{2}$$

$$\lim_{t\to 0}\frac{3}{1+t} = 3.$$

Hence,

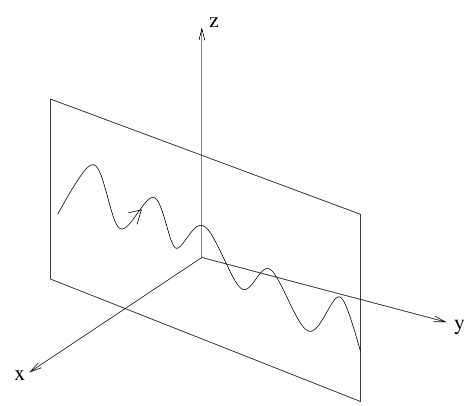
$$\lim_{t\to 0}\mathbf{r}(t) = \left\langle 1, \frac{1}{2}, 3\right\rangle.$$

Question 12. [Sec. 14.1, # 12] Sketch the curve with vector equation

$$\mathbf{r}(t) = t\,\mathbf{i} + t\,\mathbf{j} + \cos t\,\mathbf{k}.$$

Indicate with an arrow the direction in which t increases.

SOLUTION: The vector equation is equivlent to the scalar equations $x = y$ and $z = \cos t$, so that z moves along on the plane $y = x$. The curve traces out the cosine curve in the vertical plane $y = x$.

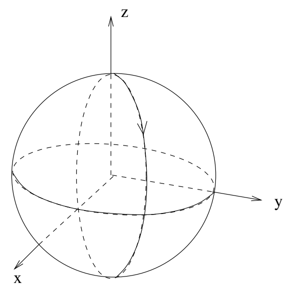


Question 13. **[Sec. 14.1, # 14]** Sketch the curve with vector equation

$$\mathbf{r}(t) = \sin t\,\mathbf{i} + \sin t\,\mathbf{j} + \sqrt{2}\cos t\,\mathbf{k}.$$

Indicate with an arrow the direction in which t increases.

SOLUTION: The vector equation of the curve is equivalent to the scalar equation $x^2 + y^2 + z^2 = \sin^2 t + \sin^2 t + 2\cos^2 t = 2(\sin^2 t + \cos^2 t) = 2$, and the curve lies on a sphere of radius $\sqrt{2}$ and center $(0, 0, 0)$. Since $x = y$, the curve is the intersection of this sphere and the plane $y = x$. So the curve is the circle of radius $\sqrt{2}$ and center $(0, 0, 0)$ in the plane $x = y$.



Question 14. **[Sec. 14.1, # 18]** Given the points $P\,(-2, 4, 0)$ and $Q\,(6, -1, 2)$, find a vector equation and parametric equations for the line segment that joins P to Q.

SOLUTION: Let $\mathbf{a} = \overrightarrow{PQ} = \langle 8, -5, 2\rangle$ be a direction vector for the line segment. Let $\mathbf{r}(t) = \langle x, y, z\rangle$, then the vector equation is:

$$\mathbf{r}(t) = \langle -2, 4, 0\rangle + t\langle 8, -5, 2\rangle, \quad 0 \le t \le 1.$$

Parametric equations are:

$$x = -2 + 8t, \quad y = 4 - 5t, \quad z = 2t, \quad 0 \le t \le 1.$$

Question 15. **[Sec. 14.1, # 34]** Find a vector function that represents the intersection of the following surfaces: the cylinder $x^2 + y^2 = 4$ and the surface $z = xy$.

SOLUTION: The projection of the curve of intersection onto the xy-plane is the circle $x^2 + y^2 = 4,\ z = 0$. So $x = 2\cos t,\ y = 2\sin t,\ 0 \le t \le 2\pi$. On the surface $z = xy$, $z = (2\cos t)(2\sin t) = 4\cos t\sin t = 2\sin 2t$. So the curve of intersection has parametric equations:

$$x = 2\cos t, \quad y = 2\sin t \quad z = 2\sin 2t, \quad 0 \le t \le 2\pi.$$

Therefore, a vector equation is

$$\mathbf{r}(t) = (2\cos t)\mathbf{i} + (2\sin t)\mathbf{j} + (2\sin 2t)\mathbf{k}, \quad 0 \le t \le 2\pi,$$

that is, $\mathbf{r}(t) = \langle 2\cos t, 2\sin t, 2\sin 2t\rangle$.