MATH 214 (R1) Winter 2008
Intermediate Calculus I

Solutions to Problem Set #7

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Question 1. [Sec. 13.1, # 8] Find the lengths of the sides of the triangle with vertices $A\,(1,2,-3)$, $B\,(3,4,-2)$, and $C\,(3,-2,1)$. Is $\triangle ABC$ a right triangle? Is it an isosceles triangle?

SOLUTION: Let $\mathbf{a} = \overrightarrow{AB} = \langle 2,2,1\rangle$, $\mathbf{b} = \overrightarrow{BC} = \langle 0,-6,3\rangle$, $\mathbf{c} = \overrightarrow{AC} = \langle 2,-4,4\rangle$, then

$$|\mathbf{a}| = \sqrt{4+4+1} = 3, \quad |\mathbf{b}| = \sqrt{36+9} = 3\sqrt{5}, \quad |\mathbf{c}| = \sqrt{4+16+16} = 6.$$

Therefore, $\triangle ABC$ is not an isosceles triangle; however, since

$$|\mathbf{a}|^2 + |\mathbf{c}|^2 = 9 + 36 = 45 = |\mathbf{b}|^2,$$

by the Pythagorean Theorem, $\triangle ABC$ is a right triangle.

Question 2. [Sec. 13.1, # 18] Show that the equation

$$4x^2 + 4y^2 + 4z^2 - 8x + 16y = 1$$

represents a sphere, and find its center and radius.

SOLUTION: We complete the square,

$$4x^2 + 4y^2 + 4z^2 - 8x + 16y = 1,$$

so that

$$4(x^2 - 2x + 1) + 4(y^2 + 4y + 4) + 4z^2 = 1 + 4 + 16,$$

that is,

$$4(x-1)^2 + 4(y+2)^2 + 4z^2 = 21,$$

and

$$(x-1)^2 + (y+2)^2 + z^2 = \frac{21}{4},$$

so the equation represents a sphere with center $(1,-2,0)$ and radius $\sqrt{21}/2$.

Question 3. [Sec. 13.1, # 20] Find an equation of a sphere if one of its diameters has enpoints $(2,1,4)$ and $(4,3,10)$.

SOLUTION: The length of the diameter is

$$\sqrt{(2-4)^2 + (1-3)^2 + (4-10)^2} = \sqrt{4+4+36} = \sqrt{44},$$

so the radius is $\sqrt{44}/2 = \sqrt{11}$. The center is the midpoint of $(2,1,4)$ and $(4,3,10)$, and

$$(h,k,l) = \left(\frac{2+4}{2}, \frac{1+3}{2}, \frac{4+10}{2}\right) = (3,2,7)$$

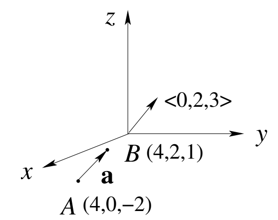
and the equation of the sphere is $(x-3)^2 + (y-2)^2 + (z-7)^2 = 11$.

Question 4. [Sec. 13.1, # 32] Describe in words the region of $\mathbb{R}^3$ represented by the equation $x^2+y^2=1$.

SOLUTION: All points on the circular cylinder with radius 1 and axis the z-axis, since the value of z is arbitrary.

Question 5. [Sec. 13.2, # 12] Find a vector **a** with representation given by the directed line seqment $\overrightarrow{AB}$ from $A\,(4,0,-2)$ to $B\,(4,2,1)$. Draw $\overrightarrow{AB}$ and the equivalent representation starting at the origin.

SOLUTION: $\mathbf{a}=\overrightarrow{AB}=\langle 0,2,3\rangle$.



Question 6. [Sec. 13.2, # 22] Let $\mathbf{a}=3\mathbf{i}-2\mathbf{k}$ and $\mathbf{b}=\mathbf{i}-\mathbf{j}+\mathbf{k}$. Find $|\mathbf{a}|$, $\mathbf{a}+\mathbf{b}$, $\mathbf{a}-\mathbf{b}$, $2\mathbf{a}$, and $3\mathbf{a}+4\mathbf{b}$.

SOLUTION: Let $\mathbf{a}=\langle 3,0,-2\rangle$, $\mathbf{b}=\langle 1,-1,1\rangle$, then

$$\begin{aligned}
|\mathbf{a}| &= \sqrt{9+4}=\sqrt{13},\\
\mathbf{a}+\mathbf{b} &= \langle 3+1,0-1,-2+1\rangle=\langle 4,-1,-1\rangle=4\mathbf{i}-\mathbf{j}-\mathbf{k},\\
\mathbf{a}-\mathbf{b} &= \langle 3-1,0+1,-2-1\rangle=\langle 2,1,-3\rangle=2\mathbf{i}+\mathbf{j}-3\mathbf{k},\\
2\mathbf{a} &= 2\,\langle 3,0,-2\rangle=\langle 6,0,-4\rangle=6\mathbf{i}-4\mathbf{k},\\
3\mathbf{a}+4\mathbf{b} &= 3\langle 3,0,-2\rangle+4\langle 1,-1,1\rangle=\langle 9,0,-6\rangle+\langle 4,-4,4\rangle\\
&= \langle 13,-4,-2\rangle=13\mathbf{i}-4\mathbf{j}-2\mathbf{k}.
\end{aligned}$$

Question 7. [Sec. 13.2, # 26] Find a vector that has the same direction as $\langle -2,4,2\rangle$ but has length 6.

SOLUTION: The unit vector in the same direction as $\mathbf{a}=\langle -2,4,2\rangle$ is

$$\frac{\mathbf{a}}{|\mathbf{a}|}=\frac{\langle -2,4,2\rangle}{\sqrt{4+16+4}}=\frac{\langle -2,4,2\rangle}{\sqrt{24}}=\frac{1}{2\sqrt{6}}\,\langle -2,4,2\rangle,$$

and to have length 6 we need

$$\frac{6\mathbf{a}}{|\mathbf{a}|}=\frac{6}{2\sqrt{6}}\,\langle -2,4,2\rangle=\frac{3}{\sqrt{6}}\,\langle -2,4,2\rangle=\langle -\sqrt{6},2\sqrt{6},\sqrt{6}\rangle.$$

Question 8. [Sec. 13.3, # 18] Find the angle between the vectors

$$\mathbf{a}=\langle 6,-3,2\rangle \qquad \text{and} \qquad \mathbf{b}=\langle 2,1,-2\rangle.$$

(First find an exact expression and then approximate to the nearest degree.)

SOLUTION: We have

$$\begin{aligned}
\cos\theta &= \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}=\frac{12-3-4}{\sqrt{36+9+4}\,\sqrt{4+1+4}}=\frac{5}{\sqrt{49}\,\sqrt{9}}=\frac{5}{7(3)}=\frac{5}{21},\\
\theta &= \cos^{-1}\left(\frac{5}{21}\right)\approx 76^\circ.
\end{aligned}$$

Question 9. [Sec. 13.3, # 24] Determine whether the given vectors are orthogonal, parallel, or neither.

(a) $\mathbf{u} = \langle -3, 9, 6\rangle, \quad \mathbf{v} = \langle 4, -12, -8\rangle$

(b) $\mathbf{u} = \mathbf{i} - \mathbf{j} + 2\mathbf{k}, \quad \mathbf{v} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$

(c) $\mathbf{u} = \langle a, b, c\rangle, \quad \mathbf{v} = \langle -b, a, 0\rangle$

SOLUTION:

(a) Observe that $\mathbf{u} = 3\langle -1, 3, 2\rangle$ and $\mathbf{v} = -4\langle -1, 3, 2\rangle$, so that $\mathbf{u} = -4/3\mathbf{v}$. Hence the vectors are parallel.

(b) $\mathbf{u} = \langle 1, -1, 2\rangle$ and $\mathbf{v} = \langle 2, -1, 1\rangle$, so that $\mathbf{u} \cdot \mathbf{v} = 2 + 1 + 2 = 5 \neq 0$, and the vectors are not orthogonal. There is no scalar t that satisfies $\mathbf{u} = t\mathbf{v}$,, and they are not parallel.

(c) $\mathbf{u} \cdot \mathbf{v} = -ab + ba + 0 = 0$, and the vectors are orthogonal (so they are not parallel).

Question 10. [Sec. 13.3, # 26] For what values of b are the vectors $\langle -6, b, 2\rangle$ and $\langle b, b^2, b\rangle$ orthogonal?

SOLUTION: They are orthogonal if $\langle -6, b, 2\rangle \cdot \langle b, b^2, b\rangle = 0$, that is, if and only if $-6b + b^3 + 2b = 0$, if and only if $b(b^2 - 4) = 0$, if and only if $b = 0$ or ± 2.

Question 11. [Sec. 13.3, # 38] Find the scalar and vector projections of $\mathbf{b}$ onto $\mathbf{a}$ if

$$\mathbf{a} = \langle -1, -2, 2\rangle \qquad \text{and} \qquad \mathbf{b} = \langle 3, 3, 4\rangle.$$

SOLUTION: The scalar projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\text{comp}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|} = \frac{-3 - 6 + 8}{\sqrt{1 + 4 + 4}} = \frac{-1}{3}.$$

The vector projection of $\mathbf{b}$ onto $\mathbf{a}$ is

$$\text{proj}_{\mathbf{a}}\mathbf{b} = \frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|^2}\mathbf{a} = \left(-\frac{1}{3}\right)\frac{\langle -1, -2, 2\rangle}{3} = -\frac{1}{9}\langle -1, -2, 2\rangle = \left\langle \frac{1}{9}, \frac{2}{9}, -\frac{2}{9} \right\rangle.$$