MATH 214 (R1) Winter 2008
Intermediate Calculus I

Solutions to Problem Set #6

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Department of Mathematical and Statistical Sciences
University of Alberta

Question 1. [Sec. 11.3, # 20] Identify the curve

$$r = \tan\theta \, \sec\theta$$

by finding a Cartesian equation for the curve.

SOLUTION: Since $x = r\cos\theta$, then $\cos\theta = x/r$ and $\tan\theta = y/x$, so that

$$r = \tan\theta \sec\theta = \frac{y}{x} \cdot \frac{r}{x} = \frac{y\,r}{x^2},$$

and $y = x^2$, that is, the curve is a parabola opening upward with vertex $(0, 0)$.

Question 2. [Sec. 11.3, # 26] Find the polar equation for the curve represented by the Cartesian equation

$$x^2 - y^2 = 1.$$

SOLUTION: Since $x = r\cos\theta$ and $y = r\sin\theta$, then

$$x^2 - y^2 = r^2\cos^2\theta - r^2\sin^2\theta = r^2(\cos^2\theta - \sin^2\theta) = r^2\cos 2\theta.$$

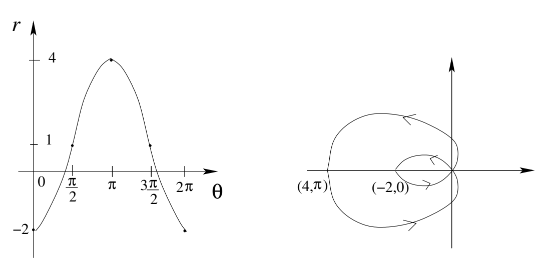
Hence the polar equation is

$$r^2\cos 2\theta = 1 \qquad \text{or} \qquad r^2 = \sec 2\theta.$$

Question 3. [Sec. 11.3, # 34] Sketch the curve with polar equation

$$r = 1 - 3\cos\theta.$$

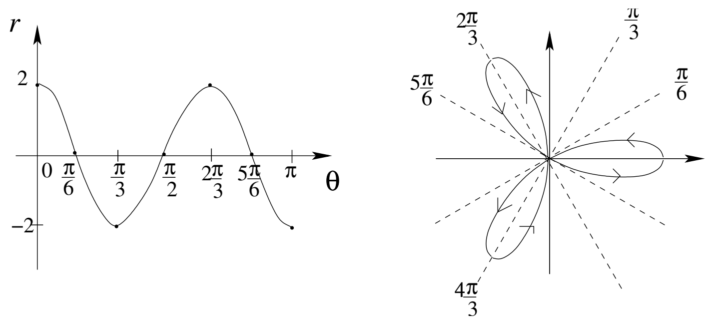
SOLUTION: The curve is sketched below.



Question 4. [Sec. 11.3, # 38] Sketch the curve with polar equation

$$r = 2\cos 3\theta.$$

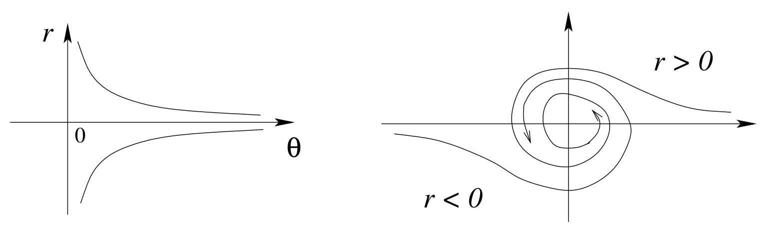
SOLUTION: Using symmetry about the polar axis, we have



Question 5. [Sec. 11.3, # 44] Sketch the curve with polar equation

$$r^2\theta = 1.$$

SOLUTION: The graph is sketched below.



Question 6. [Sec. 11.3, # 60] Find the slope of the tangent line to the polar curve

$$r = \sin 3\theta$$

at the point $\theta = \dfrac{\pi}{6}$.

SOLUTION: The slope of the tangent line is

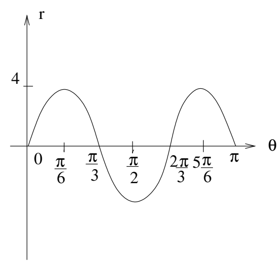
$$\frac{dy}{dx} = \frac{\dfrac{dr}{d\theta}\sin\theta + r\cos\theta}{\dfrac{dr}{d\theta}\cos\theta - r\sin\theta} = \frac{3\cos 3\theta\sin\theta + \sin 3\theta\cos\theta}{3\cos 3\theta\cos\theta - \sin 3\theta\sin\theta}$$

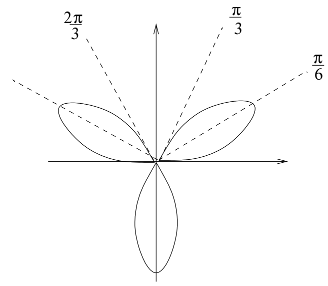
$$\left.\frac{dy}{dx}\right|_{\theta=\frac{\pi}{6}} = \frac{3\cos\frac{\pi}{2}\sin\frac{\pi}{6} + \sin\frac{\pi}{2}\cos\frac{\pi}{6}}{3\cos\frac{\pi}{2}\cos\frac{\pi}{6} - \sin\frac{\pi}{2}\sin\frac{\pi}{6}} = \frac{0 + \frac{\sqrt{3}}{2}}{0 - \frac{1}{2}} = -\sqrt{3}.$$

Question 7. [Sec. 11.4, # 18] Find the area of the region enclosed by one loop of the curve

$$r = 4\sin 3\theta.$$

Solution: The curve is sketched below.





Using symmetry we have

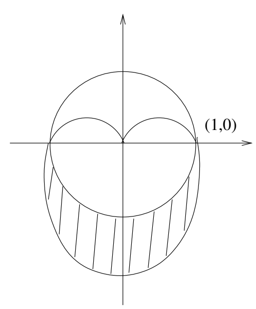
$$A = \int_0^{\frac{\pi}{3}} \frac{1}{2}(4\sin 3\theta)^2\,d\theta = \frac{1}{2}\int_0^{\frac{\pi}{6}} 16\sin^2 3\theta\,d\theta$$

$$= 8\int_0^{\frac{\pi}{6}} (1-\cos 6\theta)\,d\theta = 8\big(\theta - \frac{1}{6}\sin 6\theta\big)\Big|_0^{\frac{\pi}{6}}$$

$$= 8\big(\frac{\pi}{6} - \frac{1}{6}\sin\pi\big) = \frac{4\pi}{3}.$$

Question 8. [Sec. 11.4, # 24] Find the area of the region that lies inside the curve $r = 1-\sin\theta$ and outside the curve $r = 1$.

Solution: The curve is sketched below.



Note that the points of intersection are $\theta = 0, \pi, 2\pi$ (set $1-\sin\theta = 1$).

Using symmetry again, we have

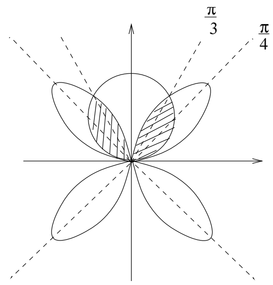
$$A = 2\cdot\frac{1}{2}\int_{-\pi/2}^{0}[(1-\sin\theta)^2 - 1^2]\,d\theta = \int_{-\frac{\pi}{2}}^{0}(-2\sin\theta + \sin^2\theta)\,d\theta$$

$$= \int_{-\frac{\pi}{2}}^{0}\big(-2\sin\theta + \frac{1}{2}(1-\cos 2\theta)\big)\,d\theta$$

$$= \Big[2\cos\theta + \frac{1}{2}\theta - \frac{1}{4}\sin 2\theta\Big]_{-\frac{\pi}{2}}^{0} = \Big(2 - \frac{1}{2}\Big(-\frac{\pi}{2}\Big)\Big) = 2 + \frac{\pi}{4}.$$

Question 9. **[Sec. 11.4, # 30]** Find the area of the region that lies inside both of the curves $r = \sin 2\theta$ and $r = \sin\theta$.

SOLUTION: The graph is sketched below.



To find the points of intersection:

$$\sin 2\theta = \sin\theta \quad \text{implies} \quad 2\sin\theta\cos\theta - \sin\theta = 0 \quad \text{implies} \quad \sin\theta(2\cos\theta - 1) = 0$$

$$\text{implies} \quad \sin\theta = 0 \quad \text{or} \quad \cos\theta = \frac{1}{2} \quad \text{implies} \quad \theta = 0, \pi, \frac{\pi}{3}, \frac{5\pi}{3}$$

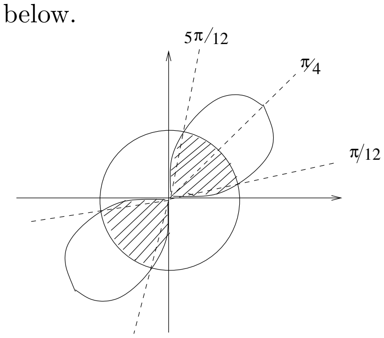
and the pole.

Using symmetry, we have

$$\begin{aligned}
A &= 2\left\{\frac{1}{2}\int_0^{\frac{\pi}{3}} \sin^2\theta\, d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^2 2\theta\, d\theta\right\} \\
&= \frac{1}{2}\int_0^{\frac{\pi}{3}} (1 - \cos 2\theta)\, d\theta + \frac{1}{2}\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} (1 - \cos 4\theta)\, d\theta \\
&= \frac{1}{2}\left\{\left[\theta - \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}} + \left[\theta - \frac{1}{4}\sin 4\theta\right]_{\frac{pi}{3}}^{\frac{\pi}{2}}\right\} \\
&= \frac{1}{2}\left\{\frac{\pi}{3} - \frac{1}{2}\sin\frac{2\pi}{3} + \frac{\pi}{2} - \frac{1}{4}\sin 2\pi - \frac{\pi}{3} + \frac{1}{4}\sin\frac{4\pi}{3}\right\} \\
&= \frac{1}{2}\left\{\frac{\pi}{2} - \frac{1}{2}\left(\frac{\sqrt{3}}{2}\right) + \frac{1}{4}\left(-\frac{\sqrt{3}}{2}\right)\right\} = \frac{1}{2}\left\{\frac{\pi}{2} - \frac{3\sqrt{3}}{8}\right\} = \frac{\pi}{4} - \frac{3\sqrt{3}}{16}.
\end{aligned}$$

Question 10. **[Sec. 11.4, # 32]** Find the area of the region that lies inside both of the curves $r^2 = 2\sin 2\theta$ and $r = 1$.

SOLUTION: The curves are sketched below.



The points of intersection:

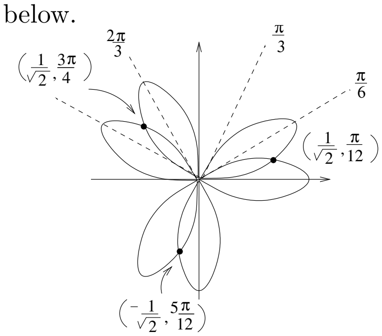
$$2\theta = \frac{\pi}{6}, \frac{5\pi}{6}, \frac{13\pi}{6}, \frac{17\pi}{6} \quad \text{implies} \quad \theta = \frac{\pi}{12}, \frac{5\pi}{12}, \frac{13\pi}{12}, \frac{17\pi}{12}.$$

Using symmetry,

$$
\begin{aligned}
A &= 2\left\{2\cdot\frac{1}{2}\int_0^{\frac{\pi}{12}} 2\sin 2\theta\,d\theta + 2\cdot\frac{1}{2}\int_{\frac{\pi}{12}}^{\frac{\pi}{4}} 1^2\,d\theta\right\}\\
&= 2\left\{\left[-\frac{2}{2}\cos 2\theta\Big|_0^{\frac{\pi}{12}} + \theta\Big|_{\frac{\pi}{12}}^{\frac{\pi}{4}}\right\} = 2\left[-\cos\frac{\pi}{6} + \cos 0 + \frac{\pi}{4} - \frac{\pi}{12}\right]\\
&= 2\left(-\frac{\sqrt{3}}{2} + 1 + \frac{2\pi}{12}\right) = 2\left(\frac{6+\pi-3\sqrt{3}}{6}\right) = 2 + \frac{\pi}{3} - \sqrt{3}.
\end{aligned}
$$

Question 11. **[Sec. 11.4, # 40]** Find all points of intersection of the curves $r = \cos 3\theta$ and $r = \sin 3\theta$.

Solution: The curves are sketched below.



Since $\cos 3\theta = \sin 3\theta$ implies that $\tan 3\theta = 1$, then $3\theta = \pi/4 + n\pi$, and thus $\theta = \pi/12 + (n\pi)/3$, and the points of intersection are

$$
\left(\frac{1}{\sqrt{2}}, \frac{\pi}{12}\right), \quad \left(\frac{1}{\sqrt{2}}, \frac{3\pi}{4}\right), \quad \left(-\frac{1}{\sqrt{2}}, \frac{5\pi}{12}\right) \quad \text{and the pole.}
$$

Question 12. **[Sec. 11.4, # 46]** Find the exact length of the of the polar curve

$$
r = e^{2\theta}, \quad 0 \le \theta \le 2\pi.
$$

Solution: The length of the curve is given by

$$
\begin{aligned}
L &= \int_0^{2\pi}\sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\,d\theta = \int_0^{2\pi}\sqrt{e^{4\theta} + (2e^{2\theta})^2}\,d\theta\\
&= \int_0^{2\pi}\sqrt{5e^{4\theta}} = \sqrt{5}\int_0^{2\pi} e^{2\theta}\,d\theta = \frac{\sqrt{5}}{2}\,e^{2\theta}\Big|_0^{2\pi} = \frac{\sqrt{5}}{2}(e^{4\pi} - 1).
\end{aligned}
$$