**MATH 214 (R1) Winter 2008**
**Intermediate Calculus I**

# Solutions to Problem Set #5

## Completion Date: Friday February 15, 2008

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**Question 1. [Sec. 11.1, # 10]** Given the parametric equations $x = t^2,\ y = t^3$

(a) Sketch the curve by using the parametric equations to plot points. Indicate with an arrow the direction in which the curve is traced as $t$ increases.

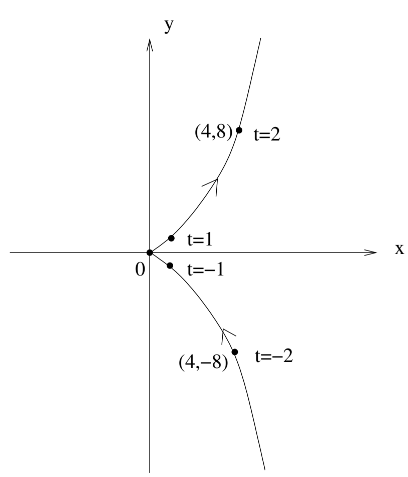
(b) Eliminate the parameter to find a Cartesian equation of the curve.

SOLUTION:

(a) Plotting points for integer values of $t$ with $-3 \leq t \leq 3$, we have

| $t$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ |
|---|---|---|---|---|---|---|---|
| $x$ | $9$ | $4$ | $1$ | $0$ | $1$ | $4$ | $9$ |
| $y$ | $-27$ | $-8$ | $-1$ | $0$ | $1$ | $8$ | $27$ |

and the curve is plotted below.




(b) Solving for the parameter $t$ in the first equation, we have $t = \pm\sqrt{x}$, so that $y = \pm x^{3/2},\ x \geq 0,\ y \in \mathbb{R}$. The Cartesian equation of the curve is given by

$$x = y^{2/3},\ x \geq 0,\ y \in \mathbb{R}.$$

**Question 2.** **[Sec. 11.1, # 12]** Given the parametric equations

$$x = 4\cos\theta, \quad y = 5\sin\theta, \quad -\pi/2 \le \theta \le \pi/2$$

(a) Eliminate the parameter to find a Cartesian equation of the curve.

(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

SOLUTION:

(a) Eliminating the parameter, since $\cos^2\theta + \sin^2\theta = 1$, we have

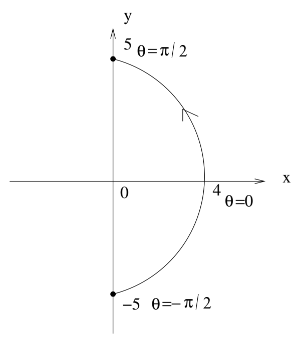
$$(x/4)^2 + (y/5)^2 = 1,$$

that is,

$$\frac{x^2}{16} + \frac{y^2}{25} = 1$$

which is the equation of an ellipse with the $x$-intercepts $x = \pm 4$, the $y$-intercepts $y = \pm 5$. However, since $-\pi/2 \le \theta \le \pi/2$, we have $0 \le \cos\theta \le 1$ so the graph consists of only the portion on the right side of the $y$-axis.

(b) The curve is plotted below.




**Question 3.** **[Sec. 11.1, # 16]** Given the parametric equations

$$x = \ln t, \quad y = \sqrt{t}, \quad t \ge 1$$

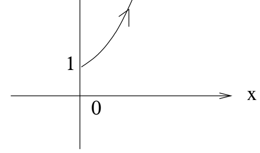
(a) Eliminate the parameter to find a Cartesian equation of the curve.

(b) Sketch the curve and indicate with an arrow the direction in which the curve is traced as the parameter increases.

SOLUTION:

(a) Eliminating the parameter, we have $x = \ln t$, which implies that $t = e^x$, so that $y = \sqrt{e^x} = e^{x/2}$, $x \ge 0$.

(b) The curve is plotted below.

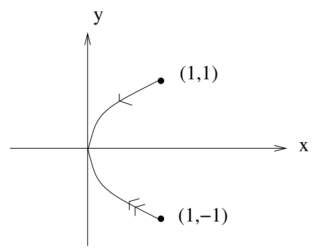


**Question 4. [Sec. 11.1, # 22]** Describe the position of a particle with position $(x, y)$ where

$$x = \cos^2 t, \quad y = \cos t, \quad 0 \le t \le 4\pi$$

as $t$ varies in the given interval.

SOLUTION: The curve $x = y^2$ is a parabola opening to the right with the vertex $(0, 0)$. The particle starts at the point $(1, 1)$ $(t = 0)$. It then moves along the parabola to the point $(0, 0)$ $(t = \pi/2)$ down to $(1, -1)$ $(t = \pi)$, then back to $(0, 0)$ $(t = 3\pi/2)$ and to $(1, 1)$ $(t = 2\pi)$. The same motion is repeated from $2\pi$ to $4\pi$.




**Question 5. [Sec. 11.2, # 8]** Find an equation of the tangent to the curve

$$x = \tan\theta, \quad y = \sec\theta$$

at the point $(1, \sqrt{2})$ by 2 methods: (a) without eliminating the parameter and (b) by first eliminating the parameter.

SOLUTION:

(a) Differentiating, we have
$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta} = \frac{\sec\theta\tan\theta}{\sec^2\theta} = \frac{\tan\theta}{\sec\theta} = \sin\theta,$$
and at $(1, \sqrt{2})$, $1 = \tan\theta$, so that $\sqrt{2} = \sec\theta$, so that $\theta = \dfrac{\pi}{4}$ or $\theta = \pi/4 + 2n\pi$, (since $\tan\theta > 0$ in quadrant I and III while $\sec\theta > 0$ in quad. I and IV). Hence the slope of the tangent at $(1, \sqrt{2})$ is

$$y'\left(\frac{\pi}{4}\right) = \sin\frac{\pi}{4} = \frac{1}{\sqrt{2}} = \frac{\sqrt{2}}{2}$$

and the equation of the tangent line is

$$y - \sqrt{2} = \frac{\sqrt{2}}{2}(x - 1).$$

(b) Since $\tan^2\theta + 1 = \sec^2\theta$ implies $x^2 + 1 = y^2$, differentiating implicitly we obtain

$$2x = 2yy'$$

which implies that
$$\frac{dy}{dx} = \frac{x}{y},$$
which in turn implies that
$$y'(1, \sqrt{2}) = \frac{1}{\sqrt{2}}$$
and this gives the same equation as in part (a).

**Question 6. [Sec. 11.2, # 16]** Find $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ if $x = \cos 2t,\ y = \cos t,\ 0 < t < \pi$. For which values of $t$ is the curve concave upward?

SOLUTION:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-\sin t}{-2\sin 2t} = \frac{\sin t}{4\sin t\cos t} = \frac{1}{4\cos t} = \frac{1}{4}\sec t,$$

$$\frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{dx/dt} = \frac{\frac{1}{4}\sec t\tan t}{-2\sin 2t} = -\frac{1}{16}\cdot\frac{\sin t}{\cos^2 t\sin t\cos t}$$
$$= -\frac{1}{16\cos^3 t}$$

or $\dfrac{dy}{dx} = -\dfrac{1}{16}\sec^3 t$. The curve is concave upward if $y''(x) > 0$ on $0 < t < \pi$, that is, if

$$-\frac{1}{16}\sec^3 t > 0,$$

or equivalently

$$\sec t < 0,$$

or

$$\cos t < 0,$$

so that

$$t \in (\pi/2, \pi).$$

**Question 7. [Sec. 11.2, # 18]** Find the points on the curve

$$x = 2t^3 + 3t^2 - 12t,\ y = 2t^3 + 3t^2 + 1$$

where the tangent is horizontal or vertical.

SOLUTION: Differentiating, we have

$$\frac{dy}{dx} = \frac{6t^2 + 6t}{6t^2 + 6t - 12} = \frac{6t(t+1)}{6(t+2)(t-1)} = \frac{t(t+1)}{(t+2)(t-1)}.$$

Horizontal tangents occur when $y' = 0$, that is, when $t = 0, -1$. Therefore the points where the tangent line if horizontal are

$$t = 0: \quad x = 0, \quad y = 1, \text{ that is, } (0,1),$$
$$t = -1: \quad x = -2 + 3 + 12 = 13, \quad y = -2 + 3 + 1 = 2, \text{ i.e., } (13,2).$$

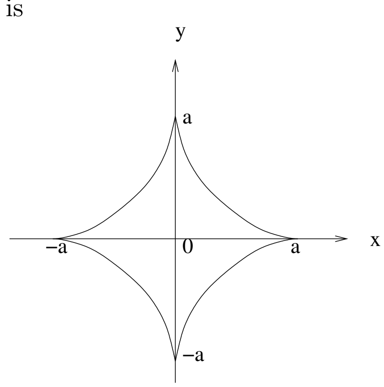
Vertical tangents occur at $t = -2, 1$ ($y'$ does not exist there), and the points where the tangent line is vertical are

$$t = -2: \quad x = -16 + 12 + 24 = 20, \quad y = -16 + 12 + 1 = -3, \text{ that is, } (20,-3),$$
$$t = 1: \quad x = 2 + 3 - 12 = -7, \quad y = 2 + 3 + 1 = 6, \text{ that is, } (-7,6).$$

**Question 8. [Sec. 11.2, # 34]** Find the area of the region enclosed by the astroid

$$x = a\cos^3\theta, \quad y = a\sin^3\theta.$$

SOLUTION: The graph of the astroid is




Since $x^{2/3} + y^{2/3} = a^{2/3}$, using symmetry we have

$$\begin{aligned}
A &= 4\int_0^a y\,dx = 4\int_0^a \left(a^{2/3} - x^{2/3}\right)^{3/2}\,dx \\
&= 4a^2\int_0^{\pi/2} 3\cos^2 t\sin^4 t\,dt = 12a^2\int_0^{\pi/2}\frac{1}{4}(2\sin t\cos t)^2\sin^2 t\,dt \\
&= \frac{3}{2}a^2\int_0^{\pi/2}\sin^2 2t(1-\cos 2t)\,dt \\
&= \frac{3}{2}a^2\left[\int_0^{\pi/2}\sin^2 2t\,dt - \int_0^{\pi/2}\sin^2 2t\cos 2t\,dt\right] \\
&= \frac{3}{2}a^2\int_0^{\pi/2}\sin^2 2t\,dt - \frac{3}{2}a^2\cdot\frac{\sin^3 2t}{6}\Bigg|_0^{\pi/2} \\
&= \frac{3a^2}{2}\int_0^{\pi/2}\left(\frac{1}{2} - \frac{1}{2}\cos 4t\right)\,dt \\
&= \frac{3\pi a^2}{8}.
\end{aligned}$$

**Question 9. [Sec. 11.2, # 44]** Find the length of the curve

$$x = e^t + e^{-t}, \quad y = 5 - 2t, \quad 0 \le t \le 3.$$

SOLUTION: Differentiating, $dx/dt = e^t - e^{-t}$ and $dy/dt = -2$, so that

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = (e^t - e^{-t})^2 + 4 = e^{2t} - 2 + e^{-2t} + 4 = e^{2t} + 2 + e^{-2t} = (e^t + e^{-t})^2$$

therefore

$$L = \int_0^3 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = \int_0^3 (e^t + e^{-t})\,dt = e^t - e^{-t}\Big|_0^3 = e^3 - e^{-3} - 1 + 1 = e^3 - e^{-3}.$$

**Question 10. [Sec. 11.2, # 60]** Find the area of the surface obtained by rotating the curve

$$x = 3t - t^3, \quad y = 3t^2, \quad 0 \le t \le 1$$

about the $x$-axis.

SOLUTION: We have

$$S = 2\pi \int_a^b y\,ds = 2\pi \int_0^1 3t^2 \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt$$

where

$$\left(\frac{dx}{dt}\right)^2 = (3 - 3t^2)^2 = 9 - 18t^2 + 9t^4$$

$$\left(\frac{dy}{dt}\right)^2 = (6t)^2 = 36t^2$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = 9 - 18t^2 + 9t^4 + 36t^2 = 9 + 18t^2 + 9t^4 = (3 + 3t^2)^2$$

therefore

$$S = 2\pi \int_0^1 3t^2(3 + 3t^2)\,dt = 2\pi \int_0^1 9t^2(1 + t^2)\,dt$$

$$= 18\pi \int_0^1 (t^2 + t^4)\,dt = 18\pi \left(\frac{t^3}{3} + \frac{t^5}{5}\right|_0^1 = 18\pi\left(\frac{1}{3} + \frac{1}{5}\right) = \frac{48\pi}{5}.$$

**Question 11. [Sec. 11.2, # 66]** Find the surface area generated by rotating the curve

$$x = e^t - t, \quad y = 4e^{t/2}, \quad 0 \le t \le 1$$

about the $y$-axis.

SOLUTION: We have $S = 2\pi \displaystyle\int_0^1 x\,ds$ where $ds = \sqrt{(dx/dt)^2 + (dy/dt)^2}\,dt$, and

$$\left(\frac{dx}{dt}\right)^2 = (e^t - 1)^2 = e^{2t} - 2e^t + 1, \quad \left(\frac{dy}{dt}\right)^2 = (2e^{t/2})^2 = 4e^t$$

therefore

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = e^{2t} - 2e^t + 1 + 4e^t = e^{2t} + 2e^t + 1 = (e^t + 1)^2$$

therefore

$$S = 2\pi \int_0^1 (e^t - t)(e^t + 1)\,dt$$

$$= 2\pi \int_0^1 (e^{2t} + e^t - te^t - t)\,dt$$

$$= 2\pi \left(\frac{1}{2}e^{2t} + e^t - (te^t - e^t) - \frac{t^2}{2}\right|_0^1$$

$$= 2\pi\left(\frac{1}{2}e^2 + e - e + e - \frac{1}{2} - \frac{1}{2} - 1 - 1\right) = \pi(e^2 + 2e - 6).$$

(Here to do $\int te^t\,dt$, integration by parts was used.)