MATH 214 (R1) Winter 2008
Intermediate Calculus I

Solutions to Problem Set #4

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Question 1. [Sec. 12.9, # 6] Find a power series representation for the function

$$f(x) = \frac{1}{1+9x^2}$$

and determine the interval of convergence.

SOLUTION: Start with the geometric series $1/(1-x) = \sum\limits_{n=0}^{\infty} x^n$, $|x| < 1$, then

$$\frac{1}{1-(-9x^2)} = \sum_{n=0}^{\infty}(-9x^2)^n = \sum_{n=0}^{\infty}(-1)^n 9^n x^{2n} = \sum_{n=0}^{\infty}(-1)^n(3x)^{2n}.$$

The series converges for $|-9x^2| < 1$, that is, for $|x^2| < 1/9$, or $|x| < 1/3$, and the interval of convergence is $(-1/3, 1/3)$.

Question 2 [Sec. 12.9, # 16] Find a power series representation for the function

$$f(x) = \frac{x^2}{(1-2x)^2}$$

and determine the radius of convergence.

SOLUTION: Start with the geometric series $1/(1-x) = \sum\limits_{n=0}^{\infty} x^n$, $|x| < 1$, then

$$\frac{1}{1-2x} = \sum_{n=0}^{\infty}(2x)^n = \sum_{n=0}^{\infty} 2^n x^n$$

converges for $|2x| < 1$, that is, $|x| < \dfrac{1}{2}$. Differentiating we obtain

$$\frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1},$$

that is,

$$\frac{1}{(1-2x)^2} = \frac{1}{2}\sum_{n=1}^{\infty} 2^n n x^{n-1} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1},$$

so that

$$\frac{x^2}{(1-2x)^2} = x^2 \sum_{n=1}^{\infty} 2^{n-1} n x^{n-1} = \sum_{n=1}^{\infty} 2^{n-1} n x^{n+1},$$

for $|x| < \dfrac{1}{2}$, and the radius of convergence is $R = 1/2$.

Question 3. [Sec. 12.9, # 18] Find a power series representation for the function

$$f(x) = \arctan(x/3)$$

and determine the radius of convergence.

SOLUTION: Recall that

$$\int \frac{1}{9+x^2}\,dx = \frac{1}{3}\tan^{-1}\frac{x}{3} + C$$

Also,

$$\frac{1}{9+x^2} = \frac{1}{9(1+(\frac{x}{3})^2)} = \frac{1}{9}\sum_{n=0}^{\infty}\left(-\left(\frac{x}{3}\right)^2\right)^n = \frac{1}{9}\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{3}\right)^{2n}$$

for $|-x^2/9| < 1$, that is, $|x/3| < 1$, or $|x| < 3$. Therefore

$$\tan^{-1}\frac{x}{3} = 3\cdot\frac{1}{9}\int\sum_{n=0}^{\infty}(-1)^n\left(\frac{x}{3}\right)^{2n}\,dx = \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\frac{1}{3^{2n}}\cdot\frac{x^{2n+1}}{2n+1} + C = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{3^{2n+1}(2n+1)} + C.$$

If we let $x = 0$, then $C = 0$, so that

$$\tan^{-1}\frac{x}{3} = \sum_{n=0}^{\infty}\frac{(-1)^n x^{2n+1}}{3^{2n+1}(2n+1)},$$

for $|x| < 3$ and the radius of convergence is $R = 3$.

Question 4. [Sec. 12.9, # 30] Use a power series representation to approximate the integral

$$f(x) = \int_0^{1/2}\frac{dx}{1+x^6}$$

to six decimal places.

SOLUTION: We use the geometric series again,

$$\int_0^{1/2}\frac{dx}{1+x^6} = \int_0^{1/2}\frac{dx}{1-(-x^6)} = \int_0^{1/2}\sum_{n=0}^{\infty}(-x^6)^n\,dx = \int_0^{1/2}\sum_{n=0}^{\infty}(-1)^n x^{6n}\,dx$$

$$= \sum_{n=0}^{\infty}(-1)^n\frac{x^{6n+1}}{6n+1}\Bigg|_0^{1/2} = \sum_{n=0}^{\infty}\frac{(-1)^n(\frac{1}{2})^{6n+1}}{6n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^{6n+1}(6n+1)}$$

$$= \frac{1}{2\cdot 1} - \frac{1}{2^7\cdot 7} + \frac{1}{2^{13}\cdot 13} - \cdots.$$

From the Alternating Series Estimation Theorem, we have $|R_n| \le b_{n+1}$, so that

$$\frac{1}{2^{13}(13)} \approx 0.0000094 \qquad \text{and} \qquad \frac{1}{2^{19}(19)} \approx 0.0000001,$$

and we use the first 3 terms of the alternating series:

$$\int_0^{1/2}\frac{dx}{1+x^6} \approx \frac{1}{2} - \frac{1}{7\cdot 2^7} + \frac{1}{13\cdot 2^{13}} \approx 0.498893.$$

Question 5. [Sec. 12.9, # 32] Show that the function

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n\, x^{2n}}{(2n)!}$$

is a solution to the differential equation

$$f''(x) + f(x) = 0.$$

SOLUTION: Differentiating,

$$f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n}{(2n)!} x^{2n-1} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-1)!} x^{2n-1},$$

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n-1)}{(2n-1)!} x^{2n-2} = \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-2)!} x^{2n-2},$$

so that

$$\begin{aligned} f''(x) + f(x) &= \sum_{n=1}^{\infty} \frac{(-1)^n}{(2n-2)!} x^{2n-2} + \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^{k+1} x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} \quad (2k = 2n-2 \text{ in the 1st sum}) \\ &= -\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} + \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} = 0. \end{aligned}$$

Question 6. [Sec. 12.10, # 4] Find the Maclaurin series for

$$f(x) = \sin 2x$$

using the definition of a Maclaurin series. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.] Also find the associated radius of convergence.

SOLUTION: We need $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \cdots$.

$$\begin{aligned} f(x) &= \sin 2x, & f(0) &= 0 \\ f'(x) &= 2\cos 2x, & f'(0) &= 2 \\ f''(x) &= -4\sin 2x, & f''(0) &= 0 \\ f'''(x) &= -8\cos 2x, & f'''(0) &= -8 = -2^3 \\ f^{(4)}(x) &= 16\sin 2x, & f^{(4)}(0) &= 0 \\ f^{(5)}(x) &= 32\cos 2x, & f^{(5)}(0) &= 32 = 2^5, \quad \text{etc,} \end{aligned}$$

therefore $f^{(n)}(0) = 0$ if n is even and $f^{(2n+1)}(0) = (-1)^n 2^{2n+1}$, and

$$f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n+1}}{(2n+1)!} x^{2n+1} = 2x - \frac{8}{3!}x^3 + \frac{32}{5!}x^5 - \cdots.$$

Also,

$$\left|\frac{a_{n+1}}{a_n}\right| = \left|\frac{(-1)^{n+1} 2^{2(n+1)+1} x^{2(n+1)+1}}{(2(n+1)+1)!} \cdot \frac{(2n+1)!}{(-1)^n 2^{2n+1} x^{2n+1}}\right| = \frac{2^2 x^2}{(2n+3)(2n+2)},$$

so that

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \frac{2^2 x^2}{(2n+3)(2n+2)} = 0 < 1$$

for all $x \in \mathbb{R}$, and by the Ratio Test $R = \infty$.

Question 7. [Sec. 12.10, # 14] Find the Taylor series for

$$f(x) = \ln x$$

centered at the value $a = 2$. [Assume that f has a power series expansion. Do not show that $R_n(x) \to 0$.]

SOLUTION: We have

$$\begin{aligned} f(x) &= \ln x & f(2) &= \ln 2 \\ f'(x) &= \frac{1}{x} & f'(2) &= \frac{1}{2} \\ f''(x) &= -\frac{1}{x^2} & f''(2) &= -\frac{1}{2^2} \\ f'''(x) &= \frac{2}{x^3} & f'''(2) &= \frac{2}{2^3} \\ f^{(4)}(x) &= -\frac{2\cdot 3}{x^4} & f^{(4)}(2) &= -\frac{2\cdot 3}{2^4}, \quad \text{etc.} \end{aligned}$$

and $f^{(n)}(2) = (-1)^{n-1}\dfrac{(n-1)!}{n}$ for $n \geq 1$, so that

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(2)}{n!}(x-2)^n = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(n-1)!}{2^n n!}(x-2)^n = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n2^n}(x-2)^n.$$

From the Ratio Test

$$\left| \frac{(-1)^n (x-2)^{n+1}}{(n+1)2^{n+1}} \cdot \frac{n2^n}{(-1)^n (x-2)^n} \right| = \frac{n|x-2|}{2(n+1)},$$

and

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n|x-2|}{2(n+1)} = \frac{1}{2}|x-2|.$$

Therefore the series converges absolutely if $\frac{1}{2}|x-2| < 1$, that is, $|x-2| < 2$, and the radius of convergence is $R = 2$.

Question 8. [Sec. 12.10, # 22] Prove that the Maclaurin series for

$$f(x) = \cosh x$$

represents $\cosh x$ for all x.

SOLUTION: First we find the Maclaurin series.

$$\begin{aligned} f(x) &= \cosh x & f(0) &= 1 \\ f'(x) &= \sinh x & f'(0) &= 0 \\ f''(x) &= \cosh x & f''(0) &= 1 \\ f'''(x) &= \sinh x & f'''(0) &= 0, \ \text{etc.} \end{aligned}$$

so the Maclaurin series is

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{f^{(2n)}(0)}{(2n)!}x^{2n} = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots$$

Now, $f^{(n+1)}(x) = \cosh x$ or $\sinh x$ and $\sinh x < \cosh x$ for all x, so that

$$|f^{(n+1)}(x)| \leq \cosh x \leq \cosh d \quad \text{if } |x| \leq d.$$

From Taylor's Inequality

$$|R_n(x)| \le \frac{M|x|^{n+1}}{(n+1)!} \le \frac{\cosh d\,|x|^{n+1}}{(n+1)!},$$

and

$$\lim_{n\to\infty} \frac{|x|^{n+1}}{(n+1)!} = 0,$$

so that

$$\lim_{n\to\infty} \frac{\cosh d\,|x|^{n+1}}{(n+1)!} = 0.$$

By the Squeeze Theorem, $\lim_{n\to\infty} R_n(x) = 0$ for $|x| \le d$, and since d is any real number, then $\cosh x$ is equal to its Maclaurin series for all x.

Question 9. [Sec. 12.10, # 30] Use a known Maclaurin series to obtain the Maclaurin series for the function

$$f(x) = \cos^2 x.$$

Hint: Use $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$.

Solution: Note that $f(x) = \frac{1}{2}(1 + \cos 2x)$, so that

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad (\text{with } R = \infty)$$

and this implies

$$\cos 2x = \sum_{n=0}^{\infty} \frac{(-1)^n (2x)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}$$

therefore

$$\cos^2 x = \frac{1}{2} + \frac{1}{2}\left\{1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n} x^{2n}}{(2n)!}\right\} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n 2^{2n-1} x^{2n}}{(2n)!},$$

for $-\infty < x < \infty$.

Question 10. [Sec. 12.10, # 46] Use series to approximate the definite integral

$$\int_0^{1/2} x^2 e^{-x^2}\, dx$$

to within the accuracy $|\text{error}| < 0.001$.

Solution: Since $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$, for $-\infty < x < \infty$, then

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} = 1 - x^2 + \frac{x^4}{2!} - \frac{x^6}{3!} + \cdots$$

$$x^2 e^{-x^2} = x^2 - x^4 + \frac{x^6}{2!} - \frac{x^8}{3!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+2}}{n!}$$

and

$$\int_0^{0.5} x^2 e^{-x^2}\,dx = \left[\frac{x^3}{3} - \frac{x^5}{5} + \frac{x^7}{7\cdot 2!} - \frac{x^9}{9\cdot 3!} + \cdots\right]_0^{0.5}$$
$$= \frac{(0.5)^3}{3} - \frac{(0.5)^5}{5} + \frac{(0.5)^7}{14} - \frac{(0.5)^9}{54} + \cdots.$$

From the Alternating Series Estimate Theorem, $|R_n| \le b_{n+1}$ and

$$\frac{(0.5)^5}{5} = 0.00625$$
$$\frac{(0.5)^7}{14} \approx 0.00056 < 0.001$$

therefore

$$\int_0^{0.5} x^2 e^{-x^2}\,dx \approx \frac{(0.5)^3}{3} - \frac{(0.5)^5}{5} \approx 0.0354.$$

Question 11. [Sec. 12.10, # 56] Find the sum of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!}.$$

Solution: We have

$$\sum_{n=0}^{\infty} \frac{(-1)^n \pi^{2n}}{6^{2n}(2n)!} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\pi}{6}\right)^{2n} = \cos\tfrac{\pi}{6} = \frac{\sqrt{3}}{2},$$

since $\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!}$.

Question 12. [Sec. 12.12, # 16a,b] Approximate

$$f(x) = \cos x$$

by a Taylor polynomial T_n with degree $n = 4$ at the number $a = \frac{\pi}{3}$, and use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the interval $0 \le x \le 2\pi/3$.

Solution:

(a) We have

$$f(x) = \cos x, \qquad f\left(\tfrac{\pi}{3}\right) = \frac{1}{2}$$
$$f'(x) = -\sin x, \qquad f'\left(\tfrac{\pi}{3}\right) = -\frac{\sqrt{3}}{2}$$
$$f''(x) = -\cos x, \qquad f''\left(\tfrac{\pi}{3}\right) = -\frac{1}{2}$$
$$f'''(x) = \sin x, \qquad f'''\left(\tfrac{\pi}{3}\right) = \frac{\sqrt{3}}{2}$$
$$f^{(4)}(x) = \cos x, \qquad f^{(4)}\left(\tfrac{\pi}{3}\right) = \frac{1}{2}$$
$$f^{(5)}(x) = -\sin x,$$

so that

$$T_4(x) = \frac{1}{2} - \frac{\sqrt{3}}{2}\left(x - \tfrac{\pi}{3}\right) - \frac{1}{2 \cdot 2!}\left(x - \tfrac{\pi}{3}\right)^2 + \frac{\sqrt{3}}{2 \cdot 3!}\left(x - \tfrac{\pi}{3}\right)^3 + \frac{1}{2 \cdot 4!}\left(x - \tfrac{\pi}{3}\right)^4.$$

(b) We have

$$|R_n(x)| \le \frac{M|x-a|^{n+1}}{(n+1)!}, \quad |x-a| \le d$$

for $|f^{(n+1)}(x)| \le M$.

If $0 \le x \le 2\pi/3$, then $|x - \pi/3| \le \pi/3$ and

$$|f^{(5)}(x)| = |-\sin x| = |\sin x| \le 1 = M$$

and therefore

$$|R_4(x)| \le \frac{1 \cdot |x - \pi/3|^5}{5!} \le \frac{(\pi/3)^5}{5!} \approx 0.01049.$$

Question 13. [Sec. 12.12, # 20a,b] Approximate

$$f(x) = x \ln x$$

by a Taylor polynomial T_n with degree $n = 3$ at the number $a = 1$, and use Taylor's Inequality to estimate the accuracy of the approximation $f(x) \approx T_n(x)$ when x lies in the interval $1/2 \le x \le 3/2$.

SOLUTION:

(a) We have

$$\begin{aligned} f(x) &= x\ln x, & f(1) &= 0 \\ f'(x) &= \ln x + 1, & f'(1) &= 1 \\ f''(x) &= \frac{1}{x}, & f''(1) &= 1 \\ f'''(x) &= -\frac{1}{x^2}, & f'''(1) &= -1 \\ f^{(4)}(x) &= \frac{2}{x^3}. \end{aligned}$$

Therefore,

$$\begin{aligned} T_3(x) &= f(1) + f'(1)(x-1) + \frac{f''(1)}{2!}(x-1)^2 + \frac{f'''(1)}{3!}(x-1)^3 \\ &= (x-1) + \frac{1}{2}(x-1)^2 - \frac{1}{3!}(x-1)^3. \end{aligned}$$

(b) We want

$$|R_3(x)| \le M|x-1|^4/4!,$$

and if $0.5 \le x \le 1.5$, then $|x - 1| \le 0.5$, and

$$(0.5)^3 \le x^3 \le (1.5)^3$$

implies

$$\frac{1}{x^3} \le \frac{1}{(0.5)^3}$$

implies

$$|f^{(4)}(x)| = \left|\frac{2}{x^3}\right| \le \frac{2}{(0.5)^3} = 16 = M$$

therefore

$$|R_3(x)| \le \frac{16(0.5)^4}{4!} \approx 0.417.$$

Question 14. [Sec. 12.12, # 26] How many terms of the Maclaurin series for $\ln(1+x)$ do you need to use to estimate $\ln 1.4$ to within 0.001?

SOLUTION: We have

$$\begin{aligned} f(x) &= \ln(1+x), & f(0) &= 0 \\ f'(x) &= \frac{1}{1+x}, & f'(0) &= 1 \\ f''(x) &= -\frac{1}{(1+x)^2}, & f''(0) &= -1 \\ f'''(x) &= \frac{2}{(1+x)^3}, & f'''(0) &= 2 \\ f^{(4)}(x) &= -\frac{6}{(1+x)^4}, & f^{(4)}(0) &= -6, \text{ etc.} \end{aligned}$$

so the Maclaurin series is

$$\ln(1+x) = (x-1) - \frac{1}{2!}(x-1)^2 + \frac{2}{3!}(x-1)^3 - \cdots$$

therefore

$$\ln(1.4) = (.4) - \frac{1}{2}(.4)^2 + \frac{1}{3}(.4)^3 - \frac{1}{4}(.4)^4 + \cdots.$$

Since this is an alternating series, we use the Alternating Series Estimation Theorem,

$$\frac{(0.4)^4}{4} = 0.0064, \quad \frac{(0.4)^5}{5} = 0.002, \quad \frac{(0.4)^6}{6} \approx 0.0007 < 0.001,$$

and we need the first 5 (nonzero) terms of the Maclaurin series.