MATH 214 (R1) Winter 2008
Intermediate Calculus I

Solutions to Problem Set #3

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Department of Mathematical and Statistical Sciences
University of Alberta

Question 1. **[Sec. 12.6, # 4]** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2^n}{n^4}$$

is absolutely convergent, conditionally convergent, or divergent.

Solution: Note that

$$\frac{2^n}{n^4} \to \infty$$

and the Test for Divergence says the series diverges.

Question 2. **[Sec. 12.6, # 8]** Determine whether the series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$$

is absolutely convergent, conditionally convergent, or divergent.

Solution: Let $b_n = n/(n^2+1) > 0$ for $n \geq 1$, then $b_n \to 0$ as $n \to \infty$ (easy) and if $f(x) = x/x^2 + 1$, then

$$f'(x) = \frac{x^2 + 1 - x \cdot 2x}{(x^2+1)^2} = \frac{1 - x^2}{(x^2+1)^2} < 0$$

if and only if $x^2 > 1$, that is, $x > 1$, and b_n is decreasing for $n \geq 2$. By the Alternating Series Test the series converges.

Now look at

$$\sum_{n=1}^{\infty} \left| (-1)^{n-1} \frac{n}{n^2+1} \right| = \sum_{n=1}^{\infty} \frac{n}{n^2+1},$$

then

$$a_n = \frac{n}{n^2+1} < \frac{n}{n^2} = \frac{1}{n} = b_n$$

and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\frac{n}{n^2+1}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1 > 0$$

and since $\sum_{n=1}^{\infty} 1/n$ diverges (the harmonic series), by the Limit Comparison Test, $\sum_{n=1}^{\infty} n/(n^2+1)$ diverges.

Therefore

$$\sum_{n=1}^{\infty} (-1)^n n/(n^2+1)$$

is conditionally convergent.

Question 3. [Sec. 12.6, # 14] Determine whether the series

$$\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2 2^n}{n!}$$

is absolutely convergent, conditionally convergent, or divergent.

SOLUTION: We use the ratio test with $a_n = \dfrac{n^2 2^n}{n!}$, we have

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2 \cdot \frac{2}{n+1} = 0 < 1,$$

and by the ratio test, the series converges absolutely.

Question 4. [Sec. 12.6, # 16] Determine whether the series

$$\sum_{n=1}^{\infty}\frac{3-\cos n}{n^{2/3}-2}$$

is absolutely convergent, conditionally convergent, or divergent.

SOLUTION: Since $-1 \le \cos n \le 1$, then $2 \le 3 - \cos n \le 4$, and

$$a_n = \frac{3-\cos n}{n^{2/3}-2} \ge \frac{2}{n^{2/3}-2} \ge \frac{2}{n^{2/3}} = b_n$$

for $n \ge 3$, (since then $n^{2/3} - 2 > 0$) and $\sum_{n=1}^{\infty} 1/n^{2/3}$ diverges (p-series with $p = 2/3 < 1$). By the Comparison Test, the given series diverges.

Question 5. [Sec. 12.6, # 18] Determine whether the series

$$\sum_{n=1}^{\infty}\frac{n!}{n^n}$$

is absolutely convergent, conditionally convergent, or divergent.

SOLUTION: We use the ratio test with $a_n = \dfrac{n!}{n^n}$, then

$$\begin{aligned}
\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| &= \lim_{n\to\infty}\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \\
&= \lim_{n\to\infty}\frac{n^n}{(n+1)^n} \\
&= \lim_{n\to\infty}\frac{1}{(1+1/n)^n} \\
&= \frac{1}{e} < 1,
\end{aligned}$$

and the given series converges by the Ratio Test.

Question 6. [Sec. 12.6, # 22] Determine whether the series

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$$

is absolutely convergent, conditionally convergent, or divergent.

SOLUTION: Note that the Ratio Test does not work since

$$\lim_{n\to\infty} \left| \frac{(-1)^{n+1}}{(n+1)\ln(n+1)} \cdot \frac{n \ln n}{(-1)^n} \right| = \lim_{n\to\infty} \frac{n}{n+1} \cdot \frac{\ln n}{\ln(n+1)} = \lim_{n\to\infty} \frac{1/n}{1/(n+1)} = \lim_{n\to\infty} \frac{n+1}{n} = 1,$$

and the Ratio test fails.

However, $b_n = 1/n \ln n > 0$ for $n \geq 2$, and since $n \ln n$ is increasing, b_n is decreasing for all $n \geq 2$. Also, $\lim\limits_{n\to\infty} 1/n \ln n = 0$ since $n \ln n \to \infty$ as $n \to \infty$. By the Alternating Series Test, the series converges.

Now,

$$\sum_{n=2}^{\infty} \left| \frac{(-1)^n}{n \ln n} \right| = \sum_{n=2}^{\infty} \frac{1}{n \ln n},$$

an if we let $f(x) = 1/x \ln x$. Then f is continuous, positive and decreasing (since $x \ln x$ is increasing) on $[2, \infty)$ and we can use the Integral Test.

$$\int_2^{\infty} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \int_2^{t} \frac{1}{x \ln x}\, dx = \lim_{t\to\infty} \int_{\ln 2}^{\ln t} \frac{1}{u}\, du \quad (u = \ln x)$$

$$= \lim_{t\to\infty} \ln u \Big|_{\ln 2}^{\ln t} = \lim_{t\to\infty} \Big[\ln(\ln t) - \ln(\ln 2) \Big] = \infty.$$

By the Integral Test, the series diverges (does not converge absolutely). Therefore, the given series is conditionally convergent.

Question 7. [Sec. 12.6, # 24] Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{(\arctan n)^n}$$

is absolutely convergent, conditionally convergent, or divergent.

SOLUTION: We use the root test, since $\tan^{-1} n > 0$ for $n \geq 1$ we have

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left| \left(\frac{-1}{\tan^{-1} n} \right)^n \right|} \lim_{n\to\infty} \frac{1}{\tan^{-1} n} = \frac{1}{\pi/2} = \frac{2}{\pi} < 1.$$

Hence by the Root Test the series converges absolutely.

Question 8. [Sec. 12.6, # 32] For which positive integers k is the series $\sum_{n=1}^{\infty} \frac{(n!)^2}{(kn)!}$ convergent?

SOLUTION: We apply the Ratio Test.

$$\left|\frac{a_{n+1}}{a_n}\right| = \frac{((n+1)!)^2}{(k(n+1))!} \cdot \frac{(kn)!}{(n!)^2} = \frac{(n+1)^2(n!)^2(kn)!}{k(n+1)\cdot(kn+k-1)\cdots(kn+1)(kn)!\cdot(n!)^2}$$

$$= \frac{(n+1)^2}{k(n+1)\cdot(kn+k-1)\cdots(kn+1)}.$$

If $k = 1$,

$$\frac{(n+1)^2}{n+1} = n+1 \to \infty \quad \text{as } n \to \infty.$$

so the series diverges.

If $k = 2$,

$$\lim_{n\to\infty} \frac{(n+1)^2}{2(n+1)(2n+1)} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{2(1+\frac{1}{n})(2+\frac{1}{n})} = \frac{1}{2(1)(2)} = \frac{1}{4} < 1$$

and the series converges absolutely.

If $k \geq 3$, then the degree of the denominator is higher than that of the numerator. Hence

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 0 < 1,$$

and therefore the series converges for $k \geq 2$.

Question 9. [Sec. 12.7, # 10] Test the series $\sum_{n=1}^{\infty} n^2 e^{-n^3}$ for convergence or divergence.

SOLUTION: Let $f(x) = x^2/e^{x^3}$. Then $f(x) > 0$ and continuous on $[1, \infty)$, and

$$f'(x) = \frac{2xe^{x^3} - x^2e^{x^3}3x^2}{(e^{x^3})^2} = \frac{xe^{x^3}(2-3x^3)}{e^{2x^3}} = \frac{x(2-3x^3)}{e^{x^3}} < 0$$

if and only if $2 - 3x^3 < 0$, that is, if and only if $3x^3 > 2$, that is, if and only if $x^3 > \frac{2}{3}$, or $x > \sqrt[3]{\frac{2}{3}}$.

Therefore f is decreasing for all $x \geq 1$ and we can use the Integral Test.

$$\int_1^{\infty} x^2 e^{-x^3}\,dx = \lim_{t\to\infty} \int_1^t x^2 e^{-x^3}\,dx = -\frac{1}{3}\lim_{t\to\infty} \int_{-1}^{-t^3} e^u\,du$$

$$= -\frac{1}{3}\lim_{t\to\infty} e^u\Big|_{-1}^{-t^3} = -\frac{1}{3}\lim_{t\to\infty}\left(e^{-t^3} - e^{-1}\right) = -\frac{1}{3}\left(-\frac{1}{e}\right) = \frac{1}{3e} < \infty$$

and the given series converges by the Integral Test.

Question 10. [Sec. 12.7, # 24] Test the series $\sum_{n=1}^{\infty} \frac{\cos(n/2)}{n^2+4n}$ for convergence or divergence.

SOLUTION: Since $|\cos(n/2)| \leq 1$, then

$$\frac{|\cos(n/2)|}{n^2+4n} \leq \frac{1}{n^2+4n} < \frac{1}{n^2}$$

and $\sum_{n=1}^{\infty} 1/n^2$ converges (p-series, $p = 2 > 1$). By the Comparison Test, the given series converges absolutely. Hence the series converges.

Question 11. [Sec. 12.7, # 28] Test the series $\sum_{n=1}^{\infty} \frac{e^{1/n}}{n^2}$ for convergence or divergence.

Solution: For $n \geq 1$,

$$\frac{1}{n} \leq 1$$

implies

$$e^{\frac{1}{n}} \leq e,$$

so that

$$\frac{e^{\frac{1}{n}}}{n^2} \leq \frac{e}{n^2}.$$

Therefore

$$\sum_{n=1}^{\infty} e/n^2 = e \sum_{n=1}^{\infty} 1/n^2$$

converges (constant multiple of a p-series with $p = 2 > 1$), and by the Comparison Test the given series converges.

Question 12. [Sec. 12.7, # 32] Test the series $\sum_{n=1}^{\infty} \frac{(2n)^n}{n^{2n}}$ for convergence or divergence.

Solution: We have

$$\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \left(\frac{2n}{n^2}\right)^n = \sum_{n=1}^{\infty} \left(\frac{2}{n}\right)^n,$$

so that

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{\left(\frac{2}{n}\right)^n} = \lim_{n\to\infty} \frac{2}{n} = 0 < 1,$$

and by the Root Test implies the given series converges.

Question 13. [Sec. 12.8, # 16] Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=0}^{\infty} n^3 (x-5)^n.$$

Solution: We use the Ratio Test to determine the radius of convergence. We have

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} = \left|\frac{(n+1)^3 (x-5)^{n+1}}{n^3 (x-5)^n}\right| = \lim_{n\to\infty} \left(\frac{n+1}{n}\right)^3 |x-5|$$

$$= \lim_{n\to\infty} \left(1 + \frac{1}{n}\right)^3 |x-5| = |x-5|,$$

therefore the series converges if $|x-5| < 1$. The radius of convergence $R = 1$, and the series converges if $-1 < x - 5 < 1$, that is, $4 < x < 6$.

To find the interval of convergence, we need to test the endpoints separately.

If $x = 4$,

$$\sum_{n=1}^{\infty} n^3 (-1)^n$$

and

$$\lim_{n\to\infty} (-1)^n n^3 = \pm\infty,$$

by the Test for Divergence, the series diverges.

If $x = 6$,

$$\sum_{n=1}^{\infty} n^3,$$

then $\lim_{n\to\infty} n^3 = \infty$, and the series diverges by the Test for Divergence.

Hence $R = 1$ and the interval of convergence is $(4, 6)$.

Question 14. [Sec. 12.8, # 20] Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{(3x-2)^n}{n3^n}.$$

SOLUTION: We have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(3x-2)^{n+1}}{(n+1)3^{n+1}} \cdot \frac{n3^n}{(3x-2)^n} \right|$$

$$\lim_{n\to\infty} \frac{n}{3(n+1)} |3x-2| = \lim_{n\to\infty} \frac{1}{3} |3x-2|.$$

By the Ratio Test, the series converges absolutely if

$$\frac{1}{3}|3x-2|,$$

that is, if

$$|3x-2| < 3,$$

that is, if

$$3\left| x - \frac{2}{3} \right| < 3,$$

or $\left| x - \frac{2}{3} \right| < 1$, so that the radius of convergence is $R = 1$. Therefore the series converges if $-1/3 < x < 5/3$.

We have to test the endpoints of the interval separately.

If $x = -1/3$, then the series becomes

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n}$$

which is the Alternating Harmonic Series, and it converges to $\log 2$.

If $x = 5/3$, then the series becomes

$$\sum_{n=1}^{\infty} 3^n/n3^n = \sum_{n=1}^{\infty} 1/n$$

which is the Harmonic series, and it diverges.

Therefore, $R = 1$ and the interval of convergence is $[-1/3, 5/3)$.

Question 15. [Sec. 12.8, # 28] Find the radius of convergence and interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdot 6 \cdot \cdots \cdot (2n)}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n-1)} x^n.$$

Solution: From the Ratio Test

$$\left| \frac{a_{n+1}}{a_n} \right| = \left| \frac{2 \cdot 4 \cdots (2n)(2n+1)(2(n+1))x^{n+1}}{1 \cdot 3 \cdots (2n-1)(2n)(2n+1)} \cdot \frac{1 \cdot 3 \cdots (2n+1)}{2 \cdot 4 \cdots (2n)x^n} \right|$$

$$= \frac{2n+2}{2n}|x| = \frac{n+1}{n}|x| \to |x| \quad \text{as } n \to \infty.$$

Therefore the series converges absolutely for $|x| < 1$ and diverges for $|x| > 1$. The radius of convergence is $R = 1$.

Again, we have to test the enpoints of the interval $-1 < x < 1$ separately.

If $x = -1$, we have

$$\sum_{n=1}^{\infty} \frac{2 \cdot 4 \cdots (2n)(-1)^n}{1 \cdot 3 \cdots (2n-1)}$$

and

$$|a_n| = \frac{2 \cdot 4 \cdots 2n}{1 \cdot 3 \cdots 2n-1} = \frac{2}{1} \cdot \frac{4}{3} \cdot \frac{6}{5} \cdots \frac{2n}{2n-1} > 1$$

since each term is greater that 1 for $n \geq 1$. Therefore $\lim_{n\to\infty} |a_n| \not\to 0$ and hence

$$\lim_{n\to\infty} a_n \not\to 0,$$

and by the Test for Divergence the series diverges when $x = -1$. The same reasoning works for $x = 1$. Hence the interval of convergence is $(-1, 1)$.