MATH 214 (R1) Winter 2008
Intermediate Calculus I

Solutions to Problem Set #2

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Question 1. [Sec. 12.4, # 8] Determine whether the series

$$\sum_{n=1}^{\infty} \frac{4+3^n}{2^n}$$

converges or diverges.

SOLUTION: Note that for $n \geq 1$

$$\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$$

and

$$\sum_{n=1}^{\infty} (3/2)^n$$

is a geometric series with $r = 3/2 > 1$ which diverges. By the Comparison Test, the given series diverges.

Question 2. [Sec. 12.4, # 10] Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n^2-1}{3n^4+1}$$

converges or diverges.

SOLUTION: For $n \geq 1$

$$\frac{n^2-1}{3n^4+1} < \frac{n^2}{3n^4+1} < \frac{n^2}{3n^4} = \frac{1}{3n^2}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

converges, since it is a p-series with $p = 2 > 1$. Therefore

$$\sum_{n=1}^{\infty} \frac{1}{3n^2}$$

converges. By the Comparison Test, the given series converges.

Question 3. [Sec. 12.4, # 12] Determine whether the series

$$\sum_{n=0}^{\infty} \frac{1+\sin n}{10^n}$$

converges or diverges.

SOLUTION: Since $-1 \leq \sin n \leq 1$ implies that $0 \leq 1 + \sin n \leq 2$,

$$0 \leq \frac{1+\sin n}{10^n} \leq \frac{2}{10^n} = 2\left(\frac{1}{10}\right)^n$$

and

$$\sum_{n=0}^{\infty} (1/10)^n$$

converges, since it is a geometric series with $0 < r = 1/10 < 1$. By the Comparison Test, the given series converges.

Question 4. [Sec. 12.4, # 20] Determine whether the series

$$\sum_{n=1}^{\infty} \frac{1+2^n}{1+3^n}$$

converges or diverges.

SOLUTION: Note that

$$a_n = \frac{1+2^n}{1+3^n} \approx \frac{2^n}{3^n} = \left(\frac{2}{3}\right)^n = b_n,$$

and

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{1+2^n}{1+3^n} \cdot \frac{3^n}{2^n} = \lim_{n\to\infty} \frac{3^n+6^n}{2^n+6^n} \\ &= \lim_{n\to\infty} \frac{\frac{3^n}{6^n}+1}{\frac{2^n}{6^n}+1} = \lim_{n\to\infty} \frac{\frac{1}{2^n}+1}{\frac{1}{3^n}+1} = \frac{1}{1} = 1 > 0\end{aligned}$$

and

$$\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} (2/3)^n$$

converges since it is a geometric series with $0 < r = 2/3 < 1$. By the Limit Comparison Test (L.C.T.), the given series converges.

Question 5. [Sec. 12.4, # 26] Determine whether the series

$$\sum_{n=1}^{\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}}$$

converges or diverges.

SOLUTION: We have

$$a_n = \frac{n+5}{\sqrt[3]{n^7+n^2}} \approx \frac{n}{\sqrt[3]{n^7}} = \frac{n}{n^{7/3}} = \frac{1}{n^{4/3}} = b_n$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n+5}{\sqrt[3]{n^7+n^2}} \cdot n^{\frac{4}{3}} = \lim_{n\to\infty} \frac{1+\frac{5n^{4/3}}{n^{7/3}}}{\sqrt[3]{1+\frac{n^2}{n^7}}} = \lim_{n\to\infty} \frac{1+\frac{5}{n}}{\sqrt[3]{1+\frac{1}{n^5}}} = 1 > 0$$

and $\sum_{n=1}^{\infty} \frac{1}{n^{4/3}}$ converges since it is a p-series with $p = 4/3 > 1$. The Limit Comparison Test implies the given series converges.

Question 6. [Sec. 12.4, # 28] Determine whether the series

$$\sum_{n=1}^{\infty} \frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)}$$

converges or diverges.

Solution: We have

$$a_n = \frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)} \approx \frac{2n^2}{3^n n^2} = \frac{2}{3^n} = 2\left(\frac{1}{3}\right)^n = b_n$$

$$\lim_{n\to\infty} \frac{2n^2 + 7n}{3^n(n^2 + 5n - 1)} \cdot \frac{3^n}{2} = \lim_{n\to\infty} \frac{2 + \frac{7}{n}}{2(1 + \frac{5}{n} - \frac{1}{n^2})} = \frac{2}{2} = 1 > 0$$

and $2\sum_{n=1}^{\infty}(1/3)^n$ converges (it's a constant multiple of a geometric series with $r = 1/3 < 1$). Therefore by the Limit Comparison Test, the given series converges.

Question 7. [Sec. 12.5, # 6] Test the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{3n - 1}$$

for convergence or divergence.

Solution: Let $b_n = 1/(3n - 1)$. Then $b_n > 0$ for $n \geq 1$ and

$$\lim_{n\to\infty} \frac{1}{3n - 1} = \lim_{n\to\infty} \frac{1/n}{3 - 1/n} = \frac{0}{3} = 0.$$

Also,

$$b_{n+1} = \frac{1}{3(n + 1) - 1} = \frac{1}{3n + 2} < \frac{1}{3n - 1} = b_n$$

and b_n is decreasing. Therefore by the Alternating Series Test, the given series is convergent.

Question 8. [Sec. 12.5, # 8] Test the series

$$\sum_{n=1}^{\infty} (-1)^n \frac{2n}{4n^2 + 1}$$

for convergence or divergence.

Solution: This is an alternating series with $b_n = 2n/(4n^2 + 1) > 0,\ \forall n \geq 1$ and

$$\lim_{n\to\infty} \frac{2n}{4n^2 + 1} = \lim_{n\to\infty} \frac{2/n}{4 + 1/n^2} = \frac{0}{4} = 0.$$

Let $f(x) = 2x/(4x^2 + 1)$, then

$$f'(x) = \frac{2(4x^2 + 1) - 2x(8x)}{(4x^2 + 1)^2} = \frac{-8x^2 + 2}{(4x^2 + 1)^2} = \frac{-2(4x^2 - 1)}{(4x^2 + 1)^2} < 0$$

and this is true if and only if $4x^2 - 1 > 0$, that is, if and only if $4x^2 > 1$, or $|x| > \dfrac{1}{2}$.

Therefore, b_n is decreasing for $n \geq 1$, and by the Alternating Series Test the given series converges.

Question 9. [Sec. 12.5, # 12] Test the series

$$\sum_{n=1}^{\infty}(-1)^{n-1}\,\frac{e^{1/n}}{n}$$

for convergence or divergence.

SOLUTION: We have $b_n = e^{1/n}/n > 0$, forall $n \geq 1$. Let $f(x) = e^{1/x}/x$. Then $\lim\limits_{x\to\infty} e^{1/x}/x = 0$ since $e^{1/x} \to 1$ and $x \to \infty$.

Also,

$$f'(x) = \frac{e^{1/x}(-\frac{1}{x^2}) - e^{1/x}}{x^2} = \frac{-e^{1/x}(\frac{1}{x^2}+1)}{x^2} = \frac{-e^{1/x}(1+x^2)}{x^4} < 0$$

for all $n \geq 1$. Thus, b_n is decreasing, and the given series converges by the Alternating Series Test.

Question 10. [Sec. 12.5, # 16] Test the series

$$\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n!}$$

for convergence or divergence.

SOLUTION: Note that

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{\sin(n\pi/2)}{n!} &= \frac{\sin\frac{\pi}{2}}{1!} + \frac{\sin\pi}{2!} + \frac{\sin\frac{3\pi}{2}}{3!} + \cdots \\ &= 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \cdots = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{(2n-1)!}\end{aligned}$$

is an alternating series with $b_n = 1/(2n-1)! > 0$.

Also,

$$b_{n+1} = \frac{1}{[2(n+1)-1]!} = \frac{1}{(2n+1)!} < \frac{1}{(2n-1)!} = b_n$$

and b_n is decreasing and $\lim\limits_{n\to\infty} 1/(2n-1)! = 0$. By the Alternating Series Test the given series converges.

Question 11. [Sec. 12.5, # 20] Test the series

$$\sum_{n=1}^{\infty}\left(-\frac{n}{5}\right)^n$$

for convergence or divergence.

SOLUTION: Note that if $f(x) = (x/5)^x$, then $y = (x/5)^x$ implies that $\ln y = x\ln(x/5)$, so that

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty} x\ln\left(\frac{x}{5}\right) = \infty,$$

and therefore $\lim\limits_{x\to\infty} f(x) = \infty$, and the Alternating Series Test cannot be applied.

However, $a_n = (-1)^n(n/5)^n \to \pm\infty$ from the above result, and the given series diverges by the Test for Divergence.

Question 12. [Sec. 12.5, # 24] How many terms of the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^4}$$

do we need to add in order to find the sum to an accuracy with $|\text{error}| < 0.001$?

SOLUTION: We first show that the alternating series converges. We have $b_n = 1/n^4 > 0$, $b_n \to 0$ as $n \to \infty$ and b_n is decreasing since n^4 is increasing, so the Alternating Series Test implies the series converges.

The remainder after n terms is $|R_n| \le b_{n+1} = 1/(n+1)^4$, and we have

$$b_4 = \frac{1}{4^4} \approx 0.0039, \qquad b_5 = \frac{1}{5^4} \approx 0.0016, \qquad b_6 = \frac{1}{6^4} \approx 0.00077 < 0.001,$$

and we need $n = 5$ terms.

Alternatively,

$$\frac{1}{(n+1)^4} < 0.001$$

implies that

$$1000 < (n+1)^4,$$

so that $n + 1 > \sqrt[4]{1000} \approx 5.62$, and this implies that $n > 4.62$, and as before we need $n = 5$ terms.

Question 13. [Sec. 12.5, # 32] For which values of p is the series

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^p}$$

convergent ?

SOLUTION: If $p > 0$, then $b_n = 1/n^p > 0$, $b_{n+1} \le b_n$ and $b_n \to 0$ as $n \to \infty$, so by the Alternating Series Test, the series converges if $p > 0$.

If $p < 0$, let $q = -p > 0$. Then

$$\lim_{n\to\infty} \frac{(-1)^{n-1}}{n^p} = \lim_{n\to\infty} (-1)^{n-1} n^{-p} = \lim_{n\to\infty} (-1)^{n-1} n^q = \pm\infty.$$

If $p = 0$, then the series becomes $\sum_{n=1}^{\infty} (-1)^{n-1}$ and

$$\lim_{n\to\infty} (-1)^{n-1} \ne 0.$$

Therefore, by the Test for Divergence, the alternating series diverges for $p \le 0$.

Hence the above series converges only for $p > 0$.