MATH 214 (R1) Winter 2008
Intermediate Calculus I

Solutions to Problem Set #10

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Question 1. [Sec. 15.2, # 8] Find the limit

$$\lim_{(x,y)\to(0,0)} \frac{x^2+\sin^2 y}{2x^2+y^2},$$

if it exists, or show that the limit does not exist.

SOLUTION: Let $(x,y)\to(0,0)$ along the x-axis ($y=0,\ x\neq 0$), then

$$f(x,y)=\frac{x^2}{2x^2}=\frac{1}{2},$$

which implies

$$\lim_{(x,y)\to(0,0)} f(x,y)=\frac{1}{2}.$$

Now let $(x,y)\to(0,0)$ along the y-axis ($x=0,\ y\neq 0$), then

$$f(x,y)=\frac{\sin^2 y}{y^2},$$

which implies

$$\lim_{(x,y)\to(0,0)}\frac{\sin^2 y}{y^2}=\left(\lim_{(x,y)\to(0,0)}\frac{\sin y}{y}\right)^2=1^2=1.$$

Since the limits along 2 different paths are not the same, $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.

Question 2. [Sec. 15.2, # 16] Find the limit

$$\lim_{(x,y)\to(0,0)}\frac{x\,y^4}{x^2+y^8},$$

if it exists, or show that the limit does not exist.

SOLUTION: Along the x-axis ($x\neq 0,\ y=0$),

$$f(x,y)=\frac{0}{x^2}=0$$

which implies

$$\lim_{(x,y)\to(0,0)} f(x,y)=0.$$

Along the path $x = y^4$,

$$f(x,y) = \frac{y^4 y^4}{y^8 + y^8} = \frac{y^8}{2y^8} = \frac{1}{2}$$

which implies

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{2}.$$

Again, the limits are different along 2 different paths so the limit of $f(x,y)$ does not exist as $(x,y) \to (0,0)$.

(Note: these are just 2 examples of paths. You may have used different paths to show the same result.)

Question 3. [Sec. 15.2, # 36] Determine the set of points at which the function

$$f(x,y) = \begin{cases} \dfrac{xy}{x^2 + xy + y^2} & \text{if} \quad (x,y) \neq (0,0) \\ 0 & \text{if} \quad (x,y) = (0,0) \end{cases}$$

is continuous.

SOLUTION: The first piece of f is a rational function that is defined for $(x,y) \neq (0,0)$ and so continuous there. We need to check continuity at $(0,0)$.

Along the y-axis $(x = 0,\ y \neq 0)$,

$$f(x,y) = \frac{0}{y^2} = 0$$

which implies

$$\lim_{(x,y)\to(0,0)} f(x,y) = 0.$$

Along the line $y = x$,

$$f(x,y) = \frac{x^2}{x^2 + x^2 + x^2} = \frac{x^2}{3x^2} = \frac{1}{3}$$

which implies

$$\lim_{(x,y)\to(0,0)} f(x,y) = \frac{1}{3}.$$

Therefore $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist (2 different limits along 2 different paths), and so f is not continuous at $(0,0)$. Thus, f is continuous on the set $\{(x,y) \,|\, (x,y) \neq (0,0)\}$.

Question 4. [Sec. 15.3, # 22] Find the first partial derivatives of the function

$$f(x,t) = \arctan\left(x\sqrt{t}\right).$$

SOLUTION: Differentiating,

$$\frac{\partial f}{\partial x} = \frac{\sqrt{t}}{1 + x^2 t}$$

$$\frac{\partial f}{\partial t} = \frac{1}{1 + x^2 t}\left(x \cdot \frac{1}{2} t^{-\frac{1}{2}}\right) = \frac{x}{2\sqrt{t}\,(1 + x^2 t)}.$$

Question 5. [Sec. 15.3, # 24] Find the first partial derivatives of the function

$$f(x,y) = \int_y^x \cos(t^2)\,dt.$$

Solution: We use the Fundamental Theorem of Calculus,

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\int_y^x \cos(t^2)\,dt = \cos(x^2)$$

$$\frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\int_y^x \cos(t^2)\,dt = -\frac{\partial}{\partial y}\int_x^y \cos(t^2)\,dt = -\cos(y^2).$$

Question 6. [Sec. 15.3, # 30] Find the first partial derivatives of the function

$$u = x^{y/z}.$$

Solution: We have

$$\frac{\partial u}{\partial x} = \frac{y}{z}\,x^{\frac{y}{z}-1},$$

$$\frac{\partial u}{\partial y} = (\ln x)x^{\frac{y}{z}}\left(\frac{1}{z}\right) = \frac{\ln x}{z}x^{\frac{y}{z}},$$

$$\frac{\partial u}{\partial z} = (\ln x)\,x^{\frac{y}{z}}\left(-\frac{y}{z^2}\right) = -\frac{y\ln x}{z^2}\,x^{\frac{y}{z}}.$$

Question 7. [Sec. 15.3, # 44] Use implicit differentiation to find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if

$$\sin(xyz) = x + 2y + 3z.$$

Solution: First we find $\dfrac{\partial z}{\partial x}$:

$$(\cos(xyz))\left(yz + xy\frac{\partial z}{\partial x}\right) = 1 + 3\frac{\partial z}{\partial x}$$

$$(xy\cos(xyz) - 3)\frac{\partial z}{\partial x} = 1 - yz\cos(xyz)$$

therefore

$$\frac{\partial z}{\partial x} = \frac{1 - yz(\cos(xyz))}{xy\cos(xyz) - 3}.$$

Next we find $\dfrac{\partial z}{\partial y}$:

$$(\cos(xyz))\left(xz + xy\frac{\partial z}{\partial y}\right) = 2 + 3\frac{\partial z}{\partial y}$$

$$(xy\cos(xyz) - 3)\frac{\partial z}{\partial y} = 2 - xz\cos(xyz)$$

therefore

$$\frac{\partial z}{\partial y} = \frac{2 - xz\cos(xyz)}{xy\cos(xyz) - 3}.$$

Question 8. [Sec. 15.3, # 46] Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ for

(a) $z = f(x)g(y)$ (b) $z = f(xy)$ (c) $z = f\big(x/y\big)$.

SOLUTION:

(a) $\dfrac{\partial z}{\partial x} = f'(x)g(y)$, $\dfrac{\partial z}{\partial y} = f(x)g'(y)$.

(b) $\dfrac{\partial z}{\partial x} = f'(xy)(y)$, $\dfrac{\partial z}{\partial y} = f'(xy)(x)$ since $u = xy$ implies that

$$\frac{\partial z}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = f'(u)(y) = yf'(xy)$$

$$\frac{\partial z}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} = f'(u)(x) = xf'(xy).$$

(c) Let $u = x/y$, then

$$\frac{\partial z}{\partial x} = \frac{df}{du}\frac{\partial u}{\partial x} = f'\left(\frac{x}{y}\right)\frac{1}{y}$$

$$\frac{\partial z}{\partial y} = \frac{df}{du}\frac{\partial u}{\partial y} = f'\left(\frac{x}{y}\right)\left(-\frac{x}{y^2}\right) = -\frac{x}{y^2}f'\left(\frac{x}{y}\right).$$

Question 9. [Sec. 15.3, # 60] Given the function

$$f(r,s,t) = r\ln(rs^2t^3),$$

find the partial derivatives f_{rss} and f_{rst}.

SOLUTION: We have

$$f_{rss} = \frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial r}\right)\right) = \frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\left(\ln(rs^2t^3) + \frac{r(s^2t^3)}{rs^2t^3}\right)\right)$$

$$= \frac{\partial}{\partial s}\left(\frac{\partial}{\partial s}\left(\ln(rs^2t^3) + 1\right)\right) = \frac{\partial}{\partial s}\left(\frac{1}{rs^2t^3}\cdot 2rst^3\right) = \frac{\partial}{\partial s}\left(\frac{2}{s}\right) = -\frac{2}{s^2}$$

$$f_{rst} = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}\left(\frac{\partial f}{\partial r}\right)\right) = \frac{\partial}{\partial t}\left(\frac{\partial}{\partial s}(\ln(rs^2t^3) + 1)\right) = \frac{\partial}{\partial t}\left(\frac{2}{s}\right) = 0.$$

Question 10. [Sec. 15.4, # 6] Find an equation of the tangent plane to the surface

$$z = e^{x^2-y^2}$$

at the point $(1,-1,1)$.

SOLUTION: The tangent plane at (x_0, y_0, z_0) has an equation of the form

$$z - z_0 = f_x(x_0,y_0)(x - x_0) + f_y(x_0,y_0)(y - y_0).$$

Note that

$$f_x = 2xe^{x^2-y^2} \quad \text{and} \quad f_y = -2ye^{x^2-y^2}.$$

At the point $(1,-1,1)$,

$$f_x(1,-1) = 2 \quad \text{and} \quad f_y(1,-1) = 2.$$

Therefore an equation of the tangent plane is

$$z - 1 = 2(x-1) + 2(y+1),$$

that is, $2x + 2y - z = -1$.

Question 11. [Sec. 15.4, # 16] Explain why the function

$$f(x, y) = \sin(2x + 3y)$$

is differentiable at the point $(-3, 2)$ and find the linearization $L(x, y)$ of the function at that point.

Solution: We have $f(x, y) = \sin(2x + 3y)$, so that

$$\begin{aligned} f_x &= 2\cos(2x + 3y) \quad \text{which implies} \quad f_x(-3, 2) = 2\cos 0 = 2 \\ f_y &= 3\cos(2x + 3y) \quad \text{which implies} \quad f_y(-3, 2) = 3\cos 0 = 3. \end{aligned}$$

Therefore, f_x and f_y exist for all (x, y) and they are continuous (since the cosine function is continuous) at $(-3, 2)$, so that f is differentiable at $(-3, 2)$, and

$$\begin{aligned} L(x, y) &= f(a, b) + f_x(a, b)(x - a) + f_y(a, b)(y - b) \\ &= f(-3, 2) + f_x(-3, 2)(x + 3) + f_y(-3, 2)(y - 2) \\ &= \sin 0 + 2(x + 3) + 3(y - 2) = 2x + 3y. \end{aligned}$$

Question 12. [Sec. 15.4, # 18] Find the linear approximation to the function

$$f(x, y, z) = \ln(x - 3y)$$

at the point $(7, 2)$ and use it to approximate $f(6.9, 2.06)$. Illustrate by graphing f and the tangent plane.

Solution: We have

$$\begin{aligned} f_x &= \frac{1}{x - 3y} \quad \text{which implies} \quad f_x(7, 2) = \frac{1}{7 - 6} = 1, \\ f_y &= \frac{-3}{x - 3y} \quad \text{which implies} \quad f_y(7, 2) = \frac{-3}{7 - 6} = -3, \end{aligned}$$

and the the linearization of f at $(7, 2)$ is

$$\begin{aligned} L(x, y) &= f(7, 2) + f_x(7, 2)(x - 7) + f_y(7, 2)(y - 2) \\ &= \ln 1 + (x - 7) - 3(y - 2) = x - 7 - 3y + 6 = x - 3y - 1, \end{aligned}$$

so that $f(x, y) \approx x - 3y - 1$. Hence

$$f(6.9, 2.06) \approx 6.9 - 3(2.06) - 1 = -.28.$$

Question 13. [Sec. 15.4, # 34] Use differentials to estimate the amount of metal in a closed cylindrical can that is 10 cm high and 4 cm in diameter if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

Solution: The volume of the can is $V = \pi r^2 h$. The amount of metal is estimated by $dV \approx \Delta V$. $\Delta h = 0.2$ cm, $\Delta r = 0.05$ cm, $r = 2$ cm and $h = 10$ cm. Therefore,

$$\begin{aligned} dV &= \frac{\partial V}{\partial r}\, dr + \frac{\partial V}{\partial h}\, dh = 2\pi r h\, dr + \pi r^2\, dh \\ &= 2\pi(2)(10)(0.05) + \pi(2^2)(0.2) = 2\pi + 0.8\pi = 2.8\pi \approx 8.8. \end{aligned}$$

The amount of metal is approximately 8.8 cm^3.

Question 14. [Sec. 15.5, # 10] Use the Chain Rule to find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ if

$$z = e^{xy} \tan y, \qquad x = s + 2t, \qquad y = s/t.$$

SOLUTION: We have

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s} = ye^{xy}\tan y + \left(\frac{1}{t}\right)(xe^{xy}\tan y + e^{xy}\sec^2 y),$$

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t} = (ye^{xy}\tan y)(2) + (xe^{xy}\tan y + e^{xy}\sec^2 y)\left(-\frac{s}{t^2}\right).$$

Question 15. [Sec. 15.5, # 14] Let

$$W(s,t) = F(u(s,t), v(s,t))$$

where F, u, and v are differentiable, and where

$$u(1,0) = 2, \quad u_s(1,0) = -2, \quad u_t(1,0) = 6$$
$$v(1,0) = 3, \quad v_s(1,0) = \ \ 5, \quad v_t(1,0) = 4$$
$$F_u(2,3) = -1, \quad F_v(2,3) = 10.$$

Find $W_s(1,0)$ and $W_t(1,0)$.

SOLUTION: Note that

$$u(1,0) = 2, \ v(1,0) = 3$$

which implies that

$$F_u(u(1,0), v(1,0)) = F_u(2,3) \qquad \text{and} \qquad F_v(u(1,0), v(1,0)) = F_v(2,3),$$

and so

$$\frac{\partial W}{\partial s} = \frac{\partial F}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial s}$$

which implies

$$\begin{aligned} W_s(1,0) &= F_u(2,3)u_s(1,0) + F_v(2,3)v_s(1,0) \\ &= -1 \cdot (-2) + (10)(5) = 2 + 50 = 52, \\ \frac{\partial W}{\partial t} &= \frac{\partial F}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial F}{\partial v}\frac{\partial v}{\partial t} \end{aligned}$$

which implies

$$\begin{aligned} W_t(1,0) &= F_u(2,3)u_t(1,0) + F_v(2,3)v_t(1,0) \\ &= (-1)(6) + (10)(4) = -6 + 40 = 34. \end{aligned}$$

Question 16. [Sec. 15.5, # 26] Let $Y = w\tan^{-1}(uv)$ where

$$u = r + s, \qquad v = s + t, \qquad w = t + r.$$

Find the partial derivatives $\dfrac{\partial Y}{\partial r}$, $\dfrac{\partial Y}{\partial s}$, and $\dfrac{\partial Y}{\partial t}$ when $r = 1,\ s = 0,\ t = 1.$

SOLUTION: From the Chain Rule we have

$$\begin{aligned}
\frac{\partial Y}{\partial r} &= \frac{\partial Y}{\partial u}\frac{\partial u}{\partial r} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial r} + \frac{\partial Y}{\partial w}\frac{\partial w}{\partial r} \\
\frac{\partial Y}{\partial s} &= \frac{\partial Y}{\partial u}\frac{\partial u}{\partial s} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial s} + \frac{\partial Y}{\partial w}\frac{\partial w}{\partial s} \\
\frac{\partial Y}{\partial t} &= \frac{\partial Y}{\partial u}\frac{\partial u}{\partial t} + \frac{\partial Y}{\partial v}\frac{\partial v}{\partial t} + \frac{\partial Y}{\partial w}\frac{\partial w}{\partial t}.
\end{aligned}$$

Therefore $r = 1,\ s = 0,\ t = 1$, we get $u = 1,\ v = 1,\ w = 2$, so that

$$\begin{aligned}
\frac{\partial Y}{\partial r} &= \left(\frac{wv}{1+u^2v^2}\right)(1) + \left(\frac{wu}{1+u^2v^2}\right)(0) + (\tan^{-1}(uv))(1) \\
&= \frac{vw}{1+u^2v^2} + \tan^{-1}(uv),
\end{aligned}$$

and

$$\left.\frac{\partial Y}{\partial r}\right|_{(1,1,2)} = \frac{2(1)}{1+1} + \tan^{-1}1 = 1 + \frac{\pi}{2}.$$

Also,

$$\begin{aligned}
\frac{\partial Y}{\partial s} &= \frac{wv}{1+u^2v^2} + \frac{wu}{1+u^2v^2} + \tan^{-1}(uv)(0) \\
&= \frac{vw}{1+u^2v^2} + \frac{uw}{1+u^2v^2},
\end{aligned}$$

and

$$\left.\frac{\partial Y}{\partial s}\right|_{(1,1,2)} = 1 + \frac{2(1)}{1+1} = 1 + 1 = 2.$$

Finally,

$$\frac{\partial Y}{\partial t} = \frac{wv}{1+u^2v^2}(0) + \frac{wu}{1+u^2v^2} + \tan^{-1}(uv)$$

and

$$\left.\frac{\partial Y}{\partial t}\right|_{(1,1,2)} = 1 + \tan^{-1}1 = 1 + \frac{\pi}{2}.$$

Question 17. [Sec. 15.5, # 30] Use the equation

$$\frac{dy}{dx} = -\frac{\dfrac{\partial F}{\partial x}}{\dfrac{\partial F}{\partial y}} = -\frac{F_x}{F_y}$$

to find $\dfrac{dy}{dx}$ if

$$\sin x + \cos y = \sin x\ \cos y.$$

SOLUTION: Let

$$F(x, y) = \sin x + \cos y - \sin x \cos y,$$

and suppose that the equation $F(x, y) = 0$ defines $y = f(x)$ as a function of x implicitly.

We have

$$F_x = \cos x - \cos x \cos y \qquad \text{and} \qquad F_y = -\sin y + \sin x \sin y,$$

so that

$$\frac{dy}{dx} = -\frac{F_x}{F_y} = \frac{\cos x(1 - \cos y)}{\sin y(1 - \sin x)}.$$