# MATH 214 (R1) Winter 2008
# Intermediate Calculus I

## Solutions to Problem Set #1

## Due: Friday January 18, 2008

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**Question 1. [Sec. 12.1, #12]** Find a formula for the general term $a_n$ of the sequence assuming that the pattern of the first few terms continues.

$$\left\{-\frac{1}{4}, \frac{2}{9}, -\frac{3}{16}, \frac{4}{25}, \dots\right\}$$

SOLUTION: We note that this is an alternating sequence where the numerator increases by 1 starting with $n = 1$ and the denominator is the square of $n + 1$ starting with $n = 1$. Hence the general formula is

$$a_n = \frac{(-1)^n\, n}{(n+1)^2}, \quad n = 1, 2, \dots.$$

**Question 2. [Sec. 12.1, #22]** Determine whether the sequence $a_n = \dfrac{(-1)^n n^3}{n^3 + 2n^2 + 1}$ converges or diverges. If it converges, find the limit.

SOLUTION: First note that

$$\lim_{n\to\infty} |a_n| = \lim_{n\to\infty} \frac{n^3}{n^3 + 2n^2 + 1} = \lim_{n\to\infty} \frac{1}{1 + \frac{2}{n} + \frac{1}{n^3}} = \frac{1}{1 + 0 + 0} = 1,$$

and the sequence $a_2, a_4, \dots$ approaches 1 whereas the sequence $a_1, a_3, \dots$ approaches $-1$ (because of the $(-1)^n$). In other words, the sequence $a_n$ oscillates between the values $-1$ and 1 as $n \to \infty$. Hence, $\lim\limits_{n\to\infty} a_n$ does not exist, and the sequence $\{a_n\}_{n\geq 1}$ diverges.

**Question 3. [Sec. 12.1, #26]** Determine whether the sequence $a_n = \arctan 2n$ converges or diverges. If it converges, find the limit.

SOLUTION: As $n \to \infty$, $2n \to \infty$, and

$$\lim_{x\to\infty} \tan^{-1} 2x = \lim_{x\to\infty} \tan^{-1} x = \frac{\pi}{2}$$

implies

$$\lim_{n\to\infty} \tan^{-1} 2n = \frac{\pi}{2},$$

so that $\{\arctan 2n\}_{n\geq 1}$ converges.

**Question 4. [Sec. 12.1, #34]** Determine whether the sequence $a_n = \sqrt{n} - \sqrt{n^2-1}$ converges or diverges. If it converges, find the limit.

SOLUTION: We calculate the limit as follows

$$\begin{aligned}
\lim_{n\to\infty}(\sqrt{n}-\sqrt{n^2-1}) &= \lim_{n\to\infty}\frac{(\sqrt{n}-\sqrt{n^2-1})(\sqrt{n}+\sqrt{n^2-1})}{\sqrt{n}+\sqrt{n^2-1}} \\
&= \lim_{n\to\infty}\frac{n-n^2+1}{\sqrt{n}+\sqrt{n^2-1}} = \lim_{n\to\infty}\frac{1-n+\frac{1}{n}}{\sqrt{\frac{1}{n}}+\sqrt{1-\frac{1}{n^2}}} = -\infty.
\end{aligned}$$

Here we divided top and bottom by $\sqrt{n^2} = n,\ n > 0$. The sequence diverges to $-\infty$.

**Question 5. [Sec. 12.1, #36]** Determine whether the sequence $a_n = \dfrac{\sin 2n}{1+\sqrt{n}}$ converges or diverges. If it converges, find the limit.

SOLUTION: Since $\sin 2n$ changes sign, we look at $|a_n|$. Also recall that $|\sin 2n| \leq 1$. Therefore

$$0 < \left|\frac{\sin 2n}{1+\sqrt{n}}\right| \leq \frac{1}{1+\sqrt{n}} < \frac{1}{\sqrt{n}} \quad \text{for } n \geq 1$$

and

$$\lim_{n\to\infty}\frac{1}{\sqrt{n}} = 0 \quad \left(\text{or} \quad \lim_{n\to\infty}\frac{1}{1+\sqrt{n}} = 0\right).$$

From the Squeeze Theorem, $|a_n| \to 0$ and therefore $a_n \to 0$ also. Therefore the sequence converges and

$$\lim_{n\to\infty}\frac{\sin 2n}{1+\sqrt{n}} = 0.$$

**Question 6. [Sec. 12.2, #20]** Determine whether the series

$$\sum_{n=1}^{\infty}\frac{e^n}{3^{n-1}}$$

is convergent or divergent. If it is convergent, find the sum.

SOLUTION: Note that this is a geometric series with $0 < r = e/3 < 1$ so converges.

$$\sum_{n=1}^{\infty}\frac{e^n}{3^{n-1}} = \sum_{n=1}^{\infty}\frac{ee^{n-1}}{3^{n-1}} = e\sum_{n=1}^{\infty}\left(\frac{e}{3}\right)^{n-1} = e\cdot\frac{1}{1-\frac{e}{3}} = \frac{3e}{3-e}.$$

(Here $0 < r = e/3 < 1$.) The sum converges.

**Question 7. [Sec. 12.2, #22]** Determine whether the series

$$\sum_{n=1}^{\infty}\frac{3}{n}$$

is convergent or divergent. If it is convergent, find the sum.

SOLUTION: This is just the harmonic series,

$$s_n = 3\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right) \to \infty$$

and the series divereges.

**Question 8. [Sec. 12.2, #24]** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{(n+1)^2}{n(n+2)}$$

is convergent or divergent. If it is convergent, find the sum.

Solution: We check $\lim_{n\to\infty} a_n$.

$$\lim_{n\to\infty} \frac{(n+1)^2}{n(n+2)} = \lim_{n\to\infty} \frac{(\frac{n+1}{n})^2}{(\frac{n+2}{n})} = \lim_{n\to\infty} \frac{(1+\frac{1}{n})^2}{1+\frac{2}{n}} = \frac{1}{1} = 1 \neq 0,$$

and by the Test for Divergence the series diverges.

**Question 9. [Sec. 12.2, #28]** Determine whether the series

$$\sum_{n=1}^{\infty} [(0.8)^{n-1} - (0.3)^n]$$

is convergent or divergent. If it is convergent, find the sum.

Solution: We note that the series looks like the difference of two geometric series.

$$\sum_{n=1}^{\infty}[(0.8)^{n-1} - (0.3)^n] = \sum_{n=1}^{\infty} \left(\frac{8}{10}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{3}{10}\right)^{n} = \sum_{n=1}^{\infty} \left(\frac{4}{5}\right)^{n-1} - \frac{3}{10}\sum_{n=1}^{\infty} \left(\frac{3}{10}\right)^{n-1}$$

so each is a convergent geometric series and

$$\sum_{n=1}^{\infty}[(0.8)^{n-1} - (0.3)^n] = \frac{1}{1-\frac{4}{5}} - \frac{3}{10} \cdot \frac{1}{1-\frac{3}{10}} = \frac{1}{\frac{1}{5}} - \frac{\frac{3}{10}}{\frac{7}{10}} = \frac{32}{7}.$$

**Question 10. [Sec. 12.2, #30]** Determine whether the series

$$\sum_{n=1}^{\infty} \ln\left(\frac{n}{2n+5}\right)$$

is convergent or divergent. If it is convergent, find the sum.

Solution: We look at the function $\ln(x/(2x+5))$.

$$\lim_{x\to\infty} \ln\left(\frac{x}{2x+5}\right) = \ln\left(\lim_{x\to\infty} \frac{x}{2x+5}\right) = \ln\frac{1}{2},$$

since ln function is continuous. Hence

$$\lim_{n\to\infty} \ln\left(\frac{n}{2n+5}\right) = \ln\frac{1}{2} \neq 0,$$

and the series diverges by the Test for Divergence.

**Question 11. [Sec. 12.2, #44]** Find the values of $x$ for which the series

$$\sum_{n=0}^{\infty} \frac{(x+3)^n}{2^n}$$

converges. Find the sum of the series for those values of $x$.

SOLUTION: The series is a geometric series and it converges iff

$$\left|\frac{x+3}{2}\right| < 1$$

that is, if and only if $|x+3| < 2$, so that $-2 < x+3 < 2$ so that $-5 < x < -1$. The sum is

$$s = \frac{1}{1 - \frac{x+3}{2}} = \frac{1}{\frac{2-x-3}{2}} = \frac{2}{-x-1} = -\frac{2}{x+1}, \qquad -5 < x < -1.$$

**Question 12. [Sec. 12.3, #10]** Determine whether the series

$$\sum_{n=1}^{\infty} (n^{-1.4} + 3n^{-1.2})$$

is convergent or divergent.

SOLUTION: We have

$$\sum_{n=1}^{\infty} (n^{-1.4} + 3n^{-1.2}) = \sum_{n=1}^{\infty} \frac{1}{n^{1.4}} + 3\sum_{n=1}^{\infty} \frac{1}{n^{1.2}}.$$

Both of these are $p$-series where $p = 1.4 > 1$ and $p = 1.2 > 1$, respectively, hence they both converge and the sum is also convergent.

**Question 13. [Sec. 12.3, #16]** Determine whether the series

$$\sum_{n=1}^{\infty} \frac{3n+2}{n(n+1)}$$

is convergent or divergent.

SOLUTION: Let $f(x) = \dfrac{3x+2}{x(x+1)}$, we find the partial fraction decomposition of the function first.

$$\frac{3x+2}{x(x+1)} = \frac{A}{x} + \frac{B}{x+1} \qquad \text{and} \qquad 3x+2 = A(x+1) + Bx,$$

therefore $3x+2 = A(x+1) + Bx$ so that $x = 0$ implies $2 = A$ and $x = -1$ implies $B = 1$. Therefore

$$f(x) = \frac{2}{x} + \frac{1}{x+1}$$

on $[1, \infty)$. It is easy to see that $f$ is continuous, positive, and decreasing since both $2/x$ and $1/(x+1)$ are decreasing hence the sum is decreasing. And

$$\int_1^{\infty} \left(\frac{2}{x} + \frac{1}{x+1}\right) dx = \lim_{t\to\infty} \int_1^{t} \left(\frac{2}{x} + \frac{1}{x+1}\right) dx = \lim_{t\to\infty} \left(2\ln x + 2\ln(x+1)\right)\Big|_1^t$$

$$= \lim_{t\to\infty} (2\ln t + 2\ln(t+1) - 2\ln 1 - 2\ln 2) = \infty,$$

and the series diverges by the Integral Test.

**Question 14.** **[Sec. 12.3, #24]** Determine whether the series

$$\sum_{n=3}^{\infty} \frac{1}{n \ln n \, \ln(\ln n)}$$

is convergent or divergent.

SOLUTION: Let $f(x) = 1/x \ln x \ln(\ln x)$ on $[3, \infty)$, it is easy to see that $f$ is continuous, positive, and decreasing since the bottom is increasing, so that

$$\begin{aligned}
\int_3^{\infty} \frac{1}{x \ln x \, \ln(\ln x)}\, dx &= \lim_{t\to\infty} \int_3^t \frac{1}{x \ln x \, \ln(\ln x)}\, dx \\
&= \lim_{t\to\infty} \int_{\ln\ln 3}^{\ln\ln t} \frac{1}{u}\, du \quad (u = \ln(\ln x),\ du = \tfrac{1}{\ln x} \cdot \tfrac{1}{x}\, dx) \\
&= \lim_{t\to\infty} \ln u \Big|_{\ln\ln 3}^{\ln\ln t} \\
&= \lim_{t\to\infty} [\ln(\ln(\ln t)) - \ln(\ln(\ln 3))] = \infty,
\end{aligned}$$

and by the Integral Test, the series diverges.

**Question 15.** **[Sec. 12.3, #32]** Find the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^5}$$

correct to three decimal places.

SOLUTION: From the Integral Test we showed that the series is convergent for $p > 1$ so the above series is convergent and the remainder term is

$$\begin{aligned}
R_n &\le \int_n^{\infty} \frac{1}{x^5}\, dx = \lim_{t\to\infty} \int_n^t \frac{1}{x^5}\, dx = \lim_{t\to\infty} -\frac{1}{4x^4}\Big|_n^t = \frac{1}{4n^4}, \\
R_1 &= \frac{1}{4} = 0.25 \\
R_2 &= \frac{1}{4 \cdot 16} = \frac{1}{64} = 0.015625 \\
R_3 &= \frac{1}{4 \cdot 3^4} = 0.003086 \\
R_4 &= \frac{1}{4 \cdot 4^4} = 0.00097656 \\
R_5 &= \frac{1}{4 \cdot 5^4} = 0.0004.
\end{aligned}$$

If we use $s_5$ to approximate the sum, then $R_5 \le 0.0004$, so

$$s \approx s_5 = 1 + \frac{1}{2^5} + \frac{1}{3^5} + \frac{1}{4^5} + \frac{1}{5^5} \approx 1.03666 \approx 1.037$$

correct to three decimal places.

**Question 16.** Show that the sequence

$$a_n = \frac{q^n}{1+q^{2n}}$$

converges to the same limit 0 for both $|q| < 1$ and $|q| > 1$.

Solution: Note that for $|q| < 1$,

$$0 \leq |a_n| = \frac{|q|^n}{1+|q|^{2n}} < |q|^n \to 0$$

as $n \to \infty$.

Also, for $|q| > 1$,

$$0 \leq |a_n| = \frac{1/|q|^n}{1+1/|q|^{2n}} < \frac{1}{|q|^n} \to 0$$

as $n \to \infty$.