MATH 214 (R1) Winter 2008
Intermediate Calculus I

Solutions to Sample Quiz Problems

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Question 1. Find the sum of the series (if possible).

(a) $\sum_{n=2}^{\infty} 3^{-n}\, 2^{n+1}$

ANSWER: $\sum_{n=2}^{\infty} 3^{-n}\, 2^{n+1} = \frac{2^3}{3^2} \sum_{n=0}^{\infty} (2/3)^n = \frac{8}{9} \frac{1}{1-2/3} = \frac{8}{3}$.

(b) $\sum_{n=0}^{\infty} \frac{4^{n+5}}{5^n}$

ANSWER: $\sum_{n=0}^{\infty} \frac{4^{n+5}}{5^n} = 4^5 \sum_{n=0}^{\infty} \frac{4^n}{5^n} = 4^5 \frac{1}{1-4/5} = 5 \cdot 4^5$.

(c) $\sum_{n=1}^{\infty} \frac{(-5)^{n+1}}{3^{n-2}}$

ANSWER: $\sum_{n=1}^{\infty} \frac{(-5)^{n+1}}{3^{n-2}}$ diverges by the Test for Divergence since $\lim_{n\to\infty} \frac{(-5)^{n+1}}{3^{n-2}}$ doesn't exist.

Question 2. Determine whether the given series is absolutely convergent, conditionally convergent, or divergent. Give all the details related to the application of appropriate convergence or divergence tests.

(a) $\sum_{n=1}^{\infty} \frac{(-1)^n}{n+\sqrt{n}}$

ANSWER: Since $a_n = \frac{1}{n+\sqrt{n}}$ is positive, decreasing, and $a_n \to 0$ as $n \to \infty$, the series converges by the alternating series test.

(b) $\sum_{n=1}^{\infty} \frac{(-100)^n}{n!}$

ANSWER: Since $a_= \frac{(100)^n}{n!}$ is positive, eventually decreasing, and $a_n \to 0$ as $n \to \infty$, the series converges by the alternating series test.

(c) $\sum_{n=1}^{\infty} \left(1 - \frac{1}{n}\right)^n$

ANSWER: Since $\lim_{n\to\infty} \left(1-\frac{1}{n}\right)^n = e^{-1} \neq 0$, the series divereges by the Test for Divergence.

Question 3. Find the radius of convergence and the interval of convergence for the power series.

(a) $\sum_{n=0}^{\infty} \frac{2^n (x-3)^n}{n+3}$

ANSWER: By the ratio test we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{2(n+3)}{n+4} |x-3| = 2|x-3| < 1$$

if $|x-3| < \frac{1}{2}$ and the radius of convergence is $R = \frac{1}{2}$.

The series converges absolutely for $\frac{5}{2} < x < \frac{7}{2}$, and at the endpoints, the series converges at $x = \frac{5}{2}$ (alternating series test) and diverges at $x = \frac{7}{2}$ (compare with harmonic series). The interval of convergence is $\left[\frac{5}{2}, \frac{7}{2}\right)$.

(b) $\sum_{n=1}^{\infty} \frac{(4x+1)^n}{n^2}$

ANSWER: By the ratio test we have

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \frac{n^2}{(n+1)^2} |4x+1| = |4x+1| < 1$$

if $|4x+1| < 1$, that is, $|x + 1/4| < 1/4$, and the radius of convergence is $R = \frac{1}{4}$.

The series converges absolutely for $-\frac{1}{2} < x < 0$, and at the endpoints, the series converges at $x = 0$ (p-series for $p = 2$) and the series converges at $x = -\frac{1}{2}$ (alternating series test). The interval of convergence is $\left[-\frac{1}{2}, 0\right]$.

Question 4. Do the following series converge? Why?

(a) $\sum_{n=1}^{\infty} \frac{10^n}{n!}$

ANSWER: The series converges to $e^{10} - 1$.

(b) $\sum_{n=1}^{\infty} \frac{\sqrt{n+1}}{n^2+n+1}$

ANSWER: The series converges by the limit comparison test (take $b_n = 1/n^{3/2}$).

(c) $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$

ANSWER: The series converges by the alternating series test.

(d) $\sum_{n=2}^{\infty} \frac{1}{(\ln n)^n}$

ANSWER: The series converges by comparison with $b_n = 1/2^n$, since $\ln n > 2$ for $n > e^2$.

(e) $\sum_{n=1}^{\infty} \frac{\cos(2n)}{n^{\frac{3}{2}}}$

ANSWER: The series converges absolutely by the comparison test (p-series with $p = 3/2$), therefore the series converges.

Question 5. A curve is given by its parametric equations: $x = 4t^2 - 5$, $y = t^3 - 3t^2 + 1$.

(a) Find $\dfrac{dy}{dx}$.

ANSWER: $\dfrac{dy}{dx} = \dfrac{dy/dt}{dx/dt} = \dfrac{3t^2 - 6t}{8t} = \dfrac{3(t-2)}{8}$ for $t \neq 0$.

(b) Find the equation of the line tangent to the curve at the point $(11, -19)$.

ANSWER: The point $(11, -19)$ on the curve occurs for the value $t = -2$ of the parameter. The slope of the tangent line at this point is

$$m = \left.\frac{dy}{dx}\right|_{t=-2} = -\frac{12}{8} = -\frac{3}{2},$$

and the point (x, y) is on the tangent line if and only if

$$y + 19 = -\frac{3}{2}(x - 11).$$

(c) At what points does the curve have horizontal tangent lines? Justify your answer.

ANSWER: The tangent line is horizontal if $\dfrac{dy}{dt} = 0$ but $\dfrac{dx}{dt} \neq 0$. For this curve, we have

$$\frac{dy}{dt} = 3t^2 - 6t = 3t(t-2) = 0$$

if and only if $t = 0$ or $t = 2$. If $t = 2$, then $\dfrac{dx}{dt} = 16$ and the tangent line is horizontal at the point $(11, -3)$.

When $t = 0$, we need to check

$$\lim_{t\to 0} \frac{3t^2 - 6t}{8t} = \lim_{t\to 0} \frac{6t - 6}{8} = -\frac{3}{4},$$

and the tangent line is not horizontal.

Question 6. Find the length of the curve:

(a) $x = e^t \cos t$, $y = e^t \sin t$, $0 \leq t \leq \pi/2$.

ANSWER: Here

$$\frac{dx}{dt} = e^t(\cos t - \sin t) \qquad \text{and} \qquad \frac{dy}{dt} = e^t(\sin t + \cos t),$$

so that

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \sqrt{2}e^t,$$

and

$$L = \int_0^{\pi/2} \sqrt{2}e^t \, dt = \left.\sqrt{2}e^t\right|_0^{\pi/2} = \sqrt{2}\left(e^{\pi/2} - 1\right).$$

(b) $x = t^3$, $y = t^2$, $0 \leq t \leq 1$

ANSWER: Here

$$\frac{dx}{dt} = 3t^2 \qquad \text{and} \qquad \frac{dy}{dt} = 2t,$$

so that

$$\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = t\sqrt{9t^2 + 4},$$

so that

$$L = \int_0^1 t\sqrt{9t^2+4}\,dt.$$

Letting $u = 9t^2 + 4$, then $du = 18tdt$, and

$$L = \frac{1}{18}\int_4^{13} u^{1/2}\,du = \frac{1}{27}u^{3/2}\Big|_4^{13} = \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right) = \frac{1}{27}\left(13\sqrt{13} - 8\right).$$

Question 7. Find the area of surface of revolution when the curve $x = t^3, \quad y = t^2, \quad 0 \le t \le 1$ is revolved about the x-axis.

ANSWER: The surface area is

$$S = 2\pi \int_0^1 y\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt = 2\pi\int_0^1 t^2\sqrt{9t^4+4t^2}\,dt = \frac{\pi}{81}\left(\frac{494}{15}\sqrt{13} + \frac{128}{15}\right).$$

Suggested problems from the text: Chapter 12 Review Problems (p. 823)

#13, 15, 16, 19, 22, 25, 29, 30, 35, 43, 45, 47, 49, 51, 57(a).