Wavelets/Framelets for Computer Graphics

The following is based on book manuscript: B. Han, Framelets and Wavelets: Algorithms, Analysis and Applications.

In this project, we only deal with computer generated curves (not surfaces). This is an easier project than the project on wavelets/framelets for signal/image processing.

To introduce a subdivision curve, we need some definitions and notation. By \( l(\mathbb{Z}) \) we denote the linear space of all sequences \( v = \{v(k)\}_{k \in \mathbb{Z}} : \mathbb{Z} \to \mathbb{C} \) of complex numbers on \( \mathbb{Z} \). One-dimensional discrete input data or signal is often treated as an element in \( l(\mathbb{Z}) \). Similarly, by \( l_0(\mathbb{Z}) \) we denote the linear space of all sequences \( u = \{u(k)\}_{k \in \mathbb{Z}} : \mathbb{Z} \to \mathbb{C} \) on \( \mathbb{Z} \) such that \( \{k \in \mathbb{Z} : u(k) \neq 0\} \) is a finite set. An element in \( l_0(\mathbb{Z}) \) is often regarded as a finite-impulse-response (FIR) filter (also called a finitely supported mask in the literature of wavelet analysis). In this book we often use \( u \) for a general filter and \( v \) for a general signal or data. It is often convenient to use the formal Fourier series (or symbol) \( \hat{v} \) of a sequence \( v = \{v(k)\}_{k \in \mathbb{Z}} \), which is defined as follows:

\[
\hat{v}(\xi) := \sum_{k \in \mathbb{Z}} v(k)e^{-ik\xi}, \quad \xi \in \mathbb{R}, \tag{1}
\]

where \( i \) in this book always denotes the imaginary unit. For \( v \in l_0(\mathbb{Z}) \), \( \hat{v} \) is a \( 2\pi \)-periodic trigonometric polynomial.

Let \( M \) be a positive integer greater than one. For a filter \( a \in l_0(\mathbb{Z}) \) and \( v \in l(\mathbb{Z}) \), the subdivision operator \( S_{M,a} : l(\mathbb{Z}) \to l(\mathbb{Z}) \) is defined to be

\[
[S_{M,a}v](n) := 2\sum_{k \in \mathbb{Z}} v(k)a(n-Mk), \quad n \in \mathbb{Z}. \tag{2}
\]

Given an initial control polygonal shape \( \{v(k)\}_{k \in \mathbb{Z}} \). We can generate a smooth curve through subdivision schemes. For \( j \in \mathbb{N} \), define

\[
v_j := S_{M,a}^{j-1}v.
\]

That is, we apply the subdivision operator \( j \) times (see the other project about how to efficiently implement a subdivision operator). Now we define “a function” \( f_j \) on the lattice \( 2^{-j}\mathbb{Z} \) as:

\[
f_j(2^{-j}k) := v_j(k), \quad k \in \mathbb{Z}.
\]

Then we connect these discrete points one-by-one to create a function \( f_j \). When \( j \to \infty \), then \( f_j \to f \), where \( f \) is the smooth subdivision curves. In practice, we only apply the subdivision scheme no more than 10 times.

To efficiently compute values \( S_{a,M}v \) on the refined reference mesh \( M^{-1}Z \) from \( v \) on the coarse mesh \( Z \), we often rewrite the subdivision operator using coset masks and convolution: For \( \beta, \gamma \in \mathbb{Z} \),

\[
[S_{a,M}v](\gamma + M\beta) = |M|\sum_{k \in \mathbb{Z}} v(k)a(\gamma + M\beta - Mk) = |M|\hat{v}[a^{[\gamma]}](\beta), \tag{3}
\]

where the coset mask \( a^{[\gamma]} \) of the mask \( a \) is defined to be

\[
a^{[\gamma]}(k) := a(\gamma + Mk), \quad k, \gamma \in \mathbb{Z}. \tag{4}
\]
If \( \sum_{k \in \mathbb{Z}} a(k) = 1 \), then \(|M| \sum_{k \in \mathbb{Z}} a(k) = 1\) for all \( \gamma \in \mathbb{Z} \). Hence, a subdivision scheme is a local averaging rule. Moreover,

\[
[S_\alpha M^\gamma](\gamma + M\beta) = |M||v \ast a(k)|(\beta) = \langle v(\beta + \cdot), |M|a(k) \rangle, \tag{5}
\]

which is attached to the point \( \beta + M^{-1} \gamma - M^{-1}c_\alpha \). Consequently, the filter

\[
\{ |M|a(k)|(-k) \}_{k \in \mathbb{Z}} = \{ |M|a(|\gamma - Mk|) \}_{k \in \mathbb{Z}}, \quad \gamma \in \{0, \ldots, M - 1\}
\]
is called the \( M^{-1}\gamma \)-stencil of the mask \( a \) for computing the cosets \( [S_\alpha M^\gamma](\gamma + M \cdot) \) on the cosets in \( M^{-1}\gamma + \mathbb{Z} \) of the refined mesh \( M^{-1}Z \). It is more convenient to use stencils for subdivision schemes in computer graphics than a filter/mask \( a \).

To deal with curves in two or three dimensions, we simply apply the subdivision scheme componentwise (that is, entrywise). Quite often we also need \( a \) to have symmetry:

\[
a(c - k) = a(k), \quad k \in \mathbb{Z}
\]

for some integer \( c \). That is, we see that \( a \) has \( \{1, -1\} \)-symmetry. For a subdivision scheme, we often use subdivision triplets: \( (a, M, \{-1, 1\}) \): \( a \) is the mask, \( M \) is the dilation factor, and \( \{-1, 1\} \) is the symmetry group. For dimension one and a dilation factor \( M \), the reference coarse mesh \( Z \) is refined into a finer mesh \( \frac{1}{M}Z \) by inserting new vertices at \( \frac{1}{M}Z + \gamma \) with \( \gamma = 1, \ldots, |M| - 1 \).

In the following, we provide a few examples of subdivision triplets.

**Example 1** \( (a, 2, \{-1, 1\}) \) is a primal subdivision triplet with

\[
a = \frac{1}{2}\{w_3, w_2, w_1, w_0, w_1, w_2, w_3\}[-3,3],
\]

where

\[
w_0 = \frac{3 + t}{4}, \quad w_1 = \frac{8 + t}{16}, \quad w_2 = \frac{1 - t}{8}, \quad w_3 = -\frac{t}{16} \quad \text{with} \quad t \in \mathbb{R}.
\]

For \( t = -\frac{1}{4} \), then \( a = a^4_0 \cdot (-3) \) and \( \text{sr}(a, 2) = 6 \), \( \text{lpm}(a) = 2 \) and \( \text{sm}(a, 2) = 5 + 1/p \) for all \( 1 \leq p \leq \infty \). \( \text{sr}(a, 2) = 4 \) if \( t \neq -1/2 \). Since \( \tilde{a}(\xi) = e^{i \xi^2}(1 + e^{-i \xi^2})^{4} \tilde{b}(\xi) \) with \( \tilde{b}(\xi) := -\frac{t}{32} + \frac{1 + 16e^{-i \xi}}{16} - \frac{1}{32}e^{-i \xi} \), by item (5) of Corollary 1, we have \( \text{sm}(a, 2) = 3 - \log_2(1 + t) \) provided \( t > -1/2 \). We only have \( \text{sm}(a, 2) \geq 3 - \log_2 |t| \) for \( t \leq -1/2 \). When \( t = 0 \), \( a = a^4_0 \cdot (-2) \) is the centered B-spline filter of order 4 with \( \text{sr}(a, 2) = 4 \) and \( \text{lpm}(a) = 2 \). When \( t = 1 \), \( a \) is an interpolatory 2-wavelet filter with \( \text{sr}(a, 2) = 4 \) and \( \text{lpm}(a) = 4 \). See Figure 1 for its subdivision stencils.

**Example 2** \( (a, 2, \{-1, 1\}) \) is a dual subdivision triplet with

\[
a = \frac{1}{2}\{w_2, w_1, w_0, w_0, w_1, w_2\}[-2,3],
\]

where

\[
w_0 = \frac{12 + 3 t}{16}, \quad w_1 = \frac{8 - 3 t}{32}, \quad w_2 = -\frac{3 t}{32} \quad \text{with} \quad t \in \mathbb{R}.
\]

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For $t = -\frac{2}{3}$, $a = a^9_1(-2)$ and $\text{sr}(a, 2) = 5$, $\text{lpm}(a) = 2$ and $\text{sm}_p(a, 2) = 4 + 1/p$ for all $1 \leq p \leq \infty$. Since $\hat{a}(\xi) = e^{2\xi}(1 + e^{-i\xi^2})^3 \hat{b}(\xi)$ with $\hat{b}(\xi) := -\frac{3t}{8} + \frac{3}{32} e^{-i\xi} - \frac{3}{8} e^{-2i\xi}$, by item (5) of Corollary ??, we have $\text{sr}(a, 2) = 3$ and $\text{sm}_\infty(a, 2) = 4 - \log_2(3tR)$ provided $t > -2/3$. We only have $\text{sm}_\infty(a, 2) \geq 1 - \log_2(3tR)$ for $t \leq -2/3$. When $t = 0$, $a = a^9_1(-1)$ is the shifted B-spline of order 3 with $\text{sr}(a, 2) = 3$ and $\text{lpm}(a) = 2$. When $t = 1$, $\text{sr}(a, 2) = 3$ and $\text{lpm}(a) = 4$. See Figure 2 for its subdivision stencils.

**Example 3** $(a, 3, \{-1, 1\})$ is a primal subdivision triplet with

$$a = \frac{1}{3} \{w_5, w_4, w_3, w_2, w_1, w_0, w_1, w_2, w_3, w_4, w_5\}_{[-5, 5]},$$

where

$$w_0 = \frac{7 - 2t_1 - 8t_2}{9}, \quad w_1 = \frac{6 - 2t_1 - 5t_2}{9}, \quad w_2 = \frac{3 + t_1 + t_2}{9}, \quad w_3 = \frac{1 + t_1 + 4t_2}{9}, \quad w_4 = \frac{t_1 + 3t_2}{9}, \quad w_5 = \frac{t_2}{9},$$

with $t_1, t_2 \in \mathbb{R}$.

For $t_1 = 2/9$ and $t_2 = 1/9$, $\text{sr}(a, 3) = 5$ and $\text{sm}_p(a, 3) = 4 + 1/p$ for all $1 \leq p \leq \infty$ whose 3-refinable function is the B-spline of order 5. Since $\hat{a}(\xi) = (e^{i\xi} + 1 + e^{-i\xi})^3 \hat{b}(\xi)$ with

$$\hat{b}(\xi) := \frac{i}{4} e^{i2\xi} + \frac{i}{4} e^{i\xi} + \frac{1 - 2t_1 - 2t_2}{2} + \frac{1}{4} e^{-i\xi} + \frac{t_1}{4} e^{-2i\xi},$$

by a similar result to item (5) of Corollary ??, we have

$$\text{sm}_\infty(a, 2) \geq 2 - \log_3 \max(\{1 - 2|t_1 - 2t_2|, 2|t_1|, 2|t_2|\}).$$

For $t_1 = 7/9$ and $t_2 = -4/9$, $a$ is an interpolatory 3-wavelet filter with $\text{sr}(a, 3) = 4 = \text{lpm}(a)$ and $\text{sm}_\infty(a, 3) \geq 4 - 1 = 3.978$. For $t_1 = 5/11$ and $t_2 = -4/11$, $a$ is an interpolatory 3-wavelet filter with $\text{sr}(a, 3) = 3 = \text{lpm}(a)$ and $\text{sm}_\infty(a, 3) \geq 2 + 1 = 3.0867$ (Using joint spectral radius, we in fact have $\text{sm}_2(a, 3) = \log_3 11 \approx 2.18266$). See Figure 3 for its subdivision stencils.

We now provide some subdivision curves in Figure 4 using the above subdivision triplets.
Figure 3: The 0-stencil (left), the $\frac{1}{3}$-stencil (middle), and $\frac{2}{3}$-stencil of the subdivision scheme in Example 3, where $w_0, \ldots, w_5$ are given in (8). Due to symmetry, the $\frac{2}{3}$-stencil is the same as the $\frac{1}{3}$-stencil. It is an interpolatory 3-wavelet filter if $w_3 = \frac{1 + r_1 + 4r_2}{9} = 0$. Since $M = 3$, each line segment (with endpoints $\circ$) is equally split into three line segments with two new inserted vertices (●) at $\frac{1}{3} + \mathbb{Z}$ and $\frac{2}{3} + \mathbb{Z}$. 
Figure 4: Subdivision curves at levels 1, 2, 3 with the initial control polygons at the first row. The subdivision triplet \((a, 2, \{-1, 1\})\) in Example 1 is used with \(t = -\frac{1}{2}\) \((aB_2(-2))\) for the 2nd row and with \(t = 1\) (interpolatory) for the 3rd row. \((a, 2, \{-1, 1\})\) in Example 2 is used with \(t = 0\) \((aB_3(-1))\), the corner cutting scheme) for the 4th row and with \(t = 1\) and \(\text{lpm}(a) = 4\) for the 5th row. \((a, 3, \{-1, 1\})\) is used with \(t_1 = \frac{5}{9}, t_2 = \frac{1}{9}\) for the 6th row and with \(t_1 = \frac{5}{11}, t_2 = -\frac{4}{11}\) (interpolatory, \(\text{sm}_m(a, 3) = \log_3 11\)) for the 7th row.