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On the role of advection in a spatial epidemic model with general boundary conditions

Kai Wang [a,b,c], Hao Wang [c], Hongyong Zhao [a,b,*]

[a] *School of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China*

[b] *Key Laboratory of Mathematical Modelling and High Performance Computing of Air Vehicles (NUAA), MIIT, Nanjing 211106, China*

[c] *Department of Mathematical and Statistical Sciences, University of Alberta, Edmonton, AB T6G 2G1, Canada*

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Abstract

This paper is concerned with a reaction-diffusion-advection model for vector-borne disease with general boundary conditions and general incidences. Due to the boundary conditions, we first apply the eigenvalue theory of elliptic system to prove the existence and uniqueness of steady state for the model. The well-posedness of the model is established using an induction argument. By overcoming the difficulty of the associated elliptic eigenvalue problem, we originally derive the variational expression of the basic reproduction ratio $\mathcal{R}_0$. The asymptotic profiles and monotonicity of $\mathcal{R}_0$ with respect to the mobility and advection rates are investigated following the variational characterization of $\mathcal{R}_0$. Furthermore, the spatial dynamics of the model with Robin type boundary condition are categorized via classifying the level set of $\mathcal{R}_0$. This work provides new clues for further research on the spread of epidemics in open advective environments.

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Keywords: Vector-borne disease; Reaction-diffusion-advection system; Basic reproduction ratio; Dynamic classification; General boundary condition

* Corresponding author at: School of Mathematics, Nanjing University of Aeronautics and Astronautics, Nanjing 211106, China.

E-mail addresses: kwang@nuaa.edu.cn (K. Wang), hao8@ualberta.ca (H. Wang), hyzho1967@126.com (H. Zhao).

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1. Introduction

Vector-borne diseases (VBDs) are illnesses caused by parasites, viruses or bacteria, and spread between people and people, people and animals, animals and animals [20]. Considering the random mobility of the host and vector, numerous reaction-diffusion compartment models were applied to probe the spread of VBDs (see, e.g., [2,29,30,33,38] and references therein). Besides random movement, in some cases, individuals can move directionally to more favorable habitats based on their own needs [3,27], or they also have biased movement in a specific direction due to the impact of external environments such as water flow and wind [28]. This process can usually be characterized by incorporating chemotactic or advection term(s) into the model. Recently, Wang et al. [28] studied a reaction-diffusion-advection vector-borne disease model with spatial heterogeneity as follows

$$\begin{cases} S_{ht} = D_1 S_{hxx} - q_h S_{hx} + H(x) - \frac{\beta_1(x) S_h f_1(x, I_v)}{S_h + I_h + R_h} - d_h(x) S_h, & 0 < x < L,\ t > 0, \\ I_{ht} = D_2 I_{hxx} - q_h I_{hx} + \frac{\beta_1(x) S_h f_1(x, I_v)}{S_h + I_h + R_h} - (d_h(x) + \gamma_h(x)) I_h, & 0 < x < L,\ t > 0, \\ R_{ht} = D_3 R_{hxx} - q_h R_{hx} + \gamma_h(x) I_h - d_h(x) R_h, & 0 < x < L,\ t > 0, \\ S_{vt} = D_4 S_{vxx} - q_v S_{vx} + V(x) - \frac{\beta_2(x) S_v f_2(x, I_h)}{S_v + I_v} - d_v(x) S_v, & 0 < x < L,\ t > 0, \\ I_{vt} = D_5 I_{vxx} - q_v I_{vx} + \frac{\beta_2(x) S_v f_2(x, I_h)}{S_v + I_v} - d_v(x) I_v, & 0 < x < L,\ t > 0, \\ D_1 S_{hx} - q_h S_h = D_2 I_{hx} - q_h I_h = D_3 R_{hx} - q_h R_h = 0, & x = 0, L,\ t > 0, \\ D_4 S_{vx} - q_v S_v = D_5 I_{vx} - q_v I_v = 0, & x = 0, L,\ t > 0, \end{cases} \tag{1.1}$$

where $S_h(x,t)$ and $I_h(x,t)$ ($S_v(x,t)$ and $I_v(x,t)$), respectively, stand for the spatial density of susceptible and infected hosts (vectors) at position x and time t in the bounded interval $[0, L]$; $R_h(x,t)$ is the spatial density of recovered hosts at x and t; L accounts for the size of habitat, and $x = 0$ and $x = L$ denote the upstream and downstream end, respectively; The diffusion rates of hosts and vectors are denoted by D_1, D_2, D_3, D_4 and D_5 respectively, and are positive; The advection rates q_h and q_v are nonnegative; The recruitment of hosts and vectors at x are represented by $H(x)$ and $V(x)$ respectively; The terms $\frac{\beta_1(x) S_h f_1(x, I_v)}{S_h + I_h + R_h}$ and $\frac{\beta_2(x) S_v f_2(x, I_h)}{S_v + I_v}$ are the infection forces; $d_h(x)$ and $d_v(x)$ are the death rates of hosts and vectors at x, respectively; The recovery rate of infected hosts is represented by $\gamma_h(x)$ at x. Other parameters of (1.1) are Hölder continuous functions in $C^{\vartheta}(\bar{\Omega})$, $\vartheta \in (0, 1)$. The authors in [28] explored the asymptotic profiles of basic reproduction ratio $\mathcal{R}_0$ of (1.1) with respect to (w.r.t) diffusion rates (D_2, D_5) and advection rates (q_h, q_v), and found that there are unique critical surfaces in the spaces $q_h - (D_2, D_5)$ and $q_v - (D_2, D_5)$ to completely separate the dynamics of (1.1) via classifying the level set of $\mathcal{R}_0$. Moreover, the aggregation phenomena of endemic equilibrium (EE) were also discussed. Nevertheless, the variational characterization of $\mathcal{R}_0$ for (1.1) has not been derived theoretically. In addition, the monotonicity of $\mathcal{R}_0$ on q_h and q_v remained unclear in [28], which is speculated that it is affected by the downstream habitat environment.

Note that model (1.1) at the upstream end $x = 0$ and downstream end $x = L$ is imposed by the no-flux type boundary condition, which means that there are no hosts and vectors across the boundaries $x = 0, L$, namely, both hosts and vectors live in a closed environment. However, the environment may be open so that individuals do not return to the habitat after leaving the downstream end due to diffusive or biased movements, resulting in loss of population [15,18]. On the other hand, susceptible and infected individuals may have distinct advection rates because of physical effects. One naturally wonders, therefore, how this open advective environment affects the extinction and persistence of VBDs. It is believed that exploring this issue not only have practical implications but also produce intriguing phenomena.

Accordingly, the above considerations lead us to propose the following model with more general boundary conditions:

$$\begin{cases} S_{ht} = D_1 S_{hxx} - a_h S_{hx} + H(x) - g_1(x, S_h, I_v) - d_h(x) S_h, & 0 < x < L,\ t > 0, \\ I_{ht} = D_2 I_{hxx} - q_h I_{hx} + g_1(x, S_h, I_v) - \delta_h(x) I_h, & 0 < x < L,\ t > 0, \\ S_{vt} = D_4 S_{vxx} - a_v S_{vx} + V(x) - g_2(x, S_v, I_h) - d_v(x) S_v, & 0 < x < L,\ t > 0, \\ I_{vt} = D_5 I_{vxx} - q_v I_{vx} + g_2(x, S_v, I_h) - d_v(x) I_v, & 0 < x < L,\ t > 0, \\ D_1 S_{hx}(0,t) - a_h S_h(0,t) = D_2 I_{hx}(0,t) - q_h I_h(0,t) = 0, & t > 0, \\ D_4 S_{vx}(0,t) - a_v S_v(0,t) = D_5 I_{vx}(0,t) - q_v I_v(0,t) = 0, & t > 0, \\ D_1 S_{hx}(L,t) - a_h S_h(L,t) = -\nu_1 a_h S_h(L,t), & t > 0, \\ D_2 I_{hx}(L,t) - q_h I_h(L,t) = -\nu_1 q_h I_h(L,t), & t > 0, \\ D_4 S_{vx}(L,t) - a_v S_v(L,t) = -\nu_2 a_v S_v(L,t), & t > 0, \\ D_5 I_{vx}(L,t) - q_v I_v(L,t) = -\nu_2 q_v I_v(L,t), & t > 0, \end{cases} \tag{1.2}$$

wherein $\delta_h(\cdot) := d_h(\cdot) + \gamma_h(\cdot)$, and $g_1(\cdot, S_h, I_v)$ and $g_2(\cdot, S_v, I_h)$ signify the disease transmission functions, a_j and q_j $(j = h, v)$ stand for the advection rates of hosts and vectors, respectively, and other parameters of model (1.2) share the same meaning as model (1.1). Since the recovered term R_h in (1.2) is decoupled from other equations, we omit it for simplicity. Inspired by the literature [18], $\nu_1 \geq 0$ and $\nu_2 \geq 0$, respectively, determine the magnitude of hosts and vectors loss at the downstream end caused by wind or water flow. The downstream biological environment can be characterized by different values of ν_1 and ν_2, which correspond mathematically to different boundary conditions at $x = L$. More specifically, for $i \in \{1, 2\}$,

(i) $\nu_i = 0$ indicates that hosts or vectors will not lose at the downstream end, which has the same boundary conditions as the upstream end. In this case, the environment is closed (see, e.g., [28]);
(ii) $0 < \nu_i < 1$ reveals that some of hosts or vectors will be lost due to wind or water flow (see, e.g., [34,42]);
(iii) $\nu_i = 1$ implies that the wind or water flow will cause the complete loss of hosts or vectors at the downstream end. In biology, this is called the "free-flow" boundary condition, which corresponds mathematically to the homogeneous Neumann type boundary condition (see, e.g., [15]);
(iv) $1 < \nu_i < \infty$ shows that both diffusive and biased movements (advection) of hosts or vectors lead to loss at the downstream end, which in fact reflects that the downstream environment is unfavorable for individual survival, and mathematically corresponds to the Robin type boundary condition (see, e.g., [36]);
(v) $\nu_i = \infty$ means that the downstream environment is extremely harsh, which mathematically can regard it as the Dirichlet type boundary condition (see, e.g., [34,42]), i.e., $S_h(L,t) = I_h(L,t) = S_v(L,t) = I_v(L,t) = 0$, $t > 0$.

Lately, there are many investigations on various boundaries in the open or closed advective environments, and readers can refer to [5–7,12,14,16,17,23,24,35] and references therein. Nevertheless, so far, very few studies focused on spatial dynamics of VBDs in advective environments. Since case (i) has been explored in [28], this work will continue to probe the remaining cases (ii)-(v). In other words, we intend to consider the system (1.2) with $\nu_1, \nu_2 \in (0, \infty)$ and the following Dirichlet type problem (i.e., $\nu_1 = \nu_2 = \infty$):

$$\begin{cases} S_{ht} = D_1 S_{hxx} - a_h S_{hx} + H(x) - g_1(x, S_h, I_v) - d_h(x) S_h, & 0 < x < L,\ t > 0, \\ I_{ht} = D_2 I_{hxx} - q_h I_{hx} + g_1(x, S_h, I_v) - \delta_h(x) I_h, & 0 < x < L,\ t > 0, \\ S_{vt} = D_4 S_{vxx} - a_v S_{vx} + V(x) - g_2(x, S_v, I_h) - d_v(x) S_v, & 0 < x < L,\ t > 0, \\ I_{vt} = D_5 I_{vxx} - q_v I_{vx} + g_2(x, S_v, I_h) - d_v(x) I_v, & 0 < x < L,\ t > 0, \\ D_1 S_{hx}(0,t) - a_h S_h(0,t) = D_2 I_{hx}(0,t) - q_h I_h(0,t) = 0, & t > 0, \\ D_4 S_{vx}(0,t) - a_v S_v(0,t) = D_5 I_{vx}(0,t) - q_v I_v(0,t) = 0, & t > 0, \\ S_h(L,t) = I_h(L,t) = S_v(L,t) = I_v(L,t) = 0, & t > 0. \end{cases} \tag{1.3}$$

To facilitate the analysis and presentation, we suppose the initial values of systems (1.2) and (1.3) satisfy

$$\begin{cases} S_h(x,0) := S_h^0(x) \ge 0,\ I_h(x,0) := I_h^0(x) \ge 0, \not\equiv 0, & 0 < x < L, \\ S_v(x,0) := S_v^0(x) \ge 0,\ I_v(x,0) := I_v^0(x) \ge 0, \not\equiv 0, & 0 < x < L, \end{cases} \tag{1.4}$$

and impose the following basic hypotheses:

- **(F1)** The $H(x)$, $V(x)$, $d_h(x)$, $d_v(x)$ and $\gamma_h(x)$ are positive in $[0, L]$; The $g_i(x, S, I)$ is positive in $C^2((0, L) \times \mathbb{R}_+ \times \mathbb{R}_+)$; $g_i(x, S, I) = 0$ if and only if $SI = 0$; $\partial_S g_i(x, S, I) > 0$, $\partial_I g_i(x, S, I) > 0$, $\partial_S^2 g_i(x, S, I) \le 0$ and $\partial_I^2 g_i(x, S, I) \le 0$ for all $x \in (0, L)$, S, $I > 0$, $i = 1, 2$.
- **(F2)** The diffusion rates (D_2, D_5) and advection rates (q_h, q_v) fulfill $\alpha := q_h/D_2 = q_v/D_5$.
- **(F3)** The death rates $(d_h(x), d_v(x))$ of (1.2) satisfy $d_h(x), d_v(x) \in C^{1+\vartheta}([0, L])$ and $d_h'(x) \ge 0$ and $d_v'(x) \ge 0$ in $[0, L]$, where $' = \mathrm{d}/\mathrm{d}x$.

Remark 1.1. Biologically, the assumption (F3) suggests that the closer the host and vector are to the downstream, the higher the mortality rate, which also indicates that the downstream environment is not conducive to individual survival.

In what follows, we state the main contributions of this paper. Due to the general incidence, general boundary condition and different diffusion coefficients, it is nontrivial to study the well-posedness of models (1.2) and (1.3). To be more specific, we first discuss the existence and uniqueness of positive steady state for an elliptic system (2.1), and then apply a well-known induction argument (see, e.g., [13,39]) to gain the well-posedness results. The previous studies on basic reproduction ratio $\mathcal{R}_0$ of vector-borne disease models with spatial heterogeneity mostly concentrated on qualitative analysis (see, e.g., [4,19]). Fortunately, the variational expressions of $\mathcal{R}_0$ for models (1.2) and (1.3) are derived with the help of the variational method [32] under suitable conditions (see Lemma 3.1). The variational formula is useful for discussing the relevant properties of $\mathcal{R}_0$, which to our knowledge seems to be the first attempt to obtain the principal eigenvalue formula of the elliptic eigenvalue problem containing two equations. More precisely, the asymptotic profiles of $\mathcal{R}_0$ in respect of diffusion rates (D_2, D_5) and advection rates (q_h, q_v) are investigated by employing the variational formula (see Proposition 4.2). Moreover, we prove the monotonicity of $\mathcal{R}_0$ w.r.t q_h and q_v in the case of $\nu_1, \nu_2 \in [1/2, \infty]$ (see Proposition 4.1). This is in sharp contrast to the results in [28], which suggests that the downstream environment has a significant impact on disease transmission. In particular, we probe the classification of dynamics of (1.2) in high- and low-risk areas, respectively. Specifically, in spaces $q_h - (D_2, D_5)$ and $q_v - (D_2, D_5)$, there are unique critical surfaces to completely separate the dynamics, that is, the disease-free equilibrium (DFE) is stable on one side of the surface and unstable on the other side, which means that the disease will disappear on one side, but break out on the other side (see Theorems 5.1 and 5.2). It should be pointed out that (i) when the habitat is a high-risk area, even if the downstream end is also located in a high-risk site (see [28, Fig. 1 (b)]), the disease will eventually be eliminated as long as the advection rates are sufficiently large relative to the diffusion rates, which is in sharp contrast to [28, Theorem 4.1]; (ii) when the habitat is a low-risk area, although the downstream end belongs to a high-risk site, there are two critical surfaces, so that the stability of DFE changes at least twice when the advection rates are within the critical surfaces, which is different from [28, Theorem 4.2 (I)].

The rest of this paper is organized as follows. Sections 2 and 3 analyze the well-posedness and threshold dynamics of systems (1.2) and (1.3). The asymptotic profiles and monotonicity of basic reproduction ratio are examined in Section 4. The spatial dynamics of (1.2) are classified in Section 5. Section 6 gives a brief discussion to conclude the article.

2. Well-posedness

Before stating the well-posedness results, we first study the DFE of (1.2) (resp. (1.3)). A solution $(\bar{S}_h(x), \bar{I}_h(x), \bar{S}_v(x), \bar{I}_v(x))$ of (1.2) (resp. (1.3)) is called the DFE if $\bar{S}_h(x), \bar{S}_v(x) > 0$ and $\bar{I}_h(x) \equiv \bar{I}_v(x) \equiv 0$ for $x \in (0, L)$, and the EE if $\bar{S}_h(x), \bar{S}_v(x), \bar{I}_h(x), \bar{I}_v(x) > 0$ for $x \in (0, L)$.

Consider two elliptic type boundary value problems as follows

$$\begin{cases} DS_{xx} - qS_x + a(x) - \mu(x)S = 0, \quad 0 < x < L, \\ DS_x(0) - qS(0) = 0, \\ DS_x(L) - qS(L) = -\nu q S(L), \end{cases} \tag{2.1}$$

and

$$\begin{cases} DS_{xx} - qS_x + a(x) - \mu(x)S = 0, \quad 0 < x < L, \\ DS_x(0) - qS(0) = 0, \ S(L) = 0, \end{cases} \tag{2.2}$$

where $D, q > 0$, $a(x) \in L^{\infty}([0, L])$, $\mu(x) \in C^{1+\vartheta}([0, L])$ and $\nu \in (0, \infty)$. Hence, there are the following conclusions.

Proposition 2.1. *Assume $a(x), \ \mu(x) > 0$, $x \in [0, L]$. Then*

(i) *If* $\mu'(x) \geq 0$, $x \in [0, L]$, *then system* (2.1) *admits a unique positive steady state* $S^*(x)$, $x \in (0, L)$;
(ii) *System* (2.2) *admits a unique positive steady state* $S^*(x)$, $x \in (0, L)$.

Proof. Consider two elliptic eigenvalue problems

$$\begin{cases} D\psi_{xx} - q\psi_x - \mu(x)\psi + \lambda\psi = 0, & 0 < x < L, \\ D\psi_x(0) - q\psi(0) = 0, \\ D\psi_x(L) - q\psi(L) = -\nu q\psi(L), \end{cases} \tag{2.3}$$

for $\nu \in (0, \infty)$, and

$$\begin{cases} D\phi_{xx} - q\phi_x - \mu(x)\phi + \kappa\phi = 0, & 0 < x < L, \\ D\phi_x(0) - q\phi(0) = 0, \ \phi(L) = 0. \end{cases} \tag{2.4}$$

Following the Krein-Rutman theorem [11] that systems (2.3) and (2.4) have principal eigenvalues λ_1 and κ_1, respectively, and the corresponding positive eigenfunctions are denoted as ψ_1 and ϕ_1, respectively. To describe more clearly, we let $\lambda_1 := \lambda_1(D, q, \mu)$ and $\kappa_1 := \kappa_1(D, q, \mu)$ to indicate the eigenvalues depend the parameters D, q and μ. Set $\psi_1 = e^{qx/D}\Psi_1$ and $\phi_1 = e^{qx/D}\Phi_1$. By a simple calculation, systems (2.3) and (2.4) are transformed into

$$\begin{cases} D\Psi_{1xx} + q\Psi_{1x} - \mu(x)\Psi_1 + \lambda_1\Psi_1 = 0, & 0 < x < L, \\ \Psi_{1x}(0) = 0, \ D\Psi_{1x}(L) + \nu q\Psi_1(L) = 0, \end{cases} \tag{2.5}$$

and

$$\begin{cases} D\Phi_{1xx} + q\Phi_{1x} - \mu(x)\Phi_1 + \kappa_1\Phi_1 = 0, & 0 < x < L, \\ \Phi_{1x}(0) = 0, \ \Phi_1(L) = 0. \end{cases} \tag{2.6}$$

According to the proof of Proposition 2.1 in [42], $\lambda_1(D, q, \mu)$ is strictly monotonically increasing function of q provided that $\mu'(x) \geq 0$. By [34, Proposition 2.1], $\kappa_1(D, q, \mu)$ is strictly monotonically increasing function of q. It is easy to see that $\lambda_1(D, 0, \mu) > 0$ owing to $\mu(\cdot) > 0$ for any $D > 0$. Moreover, one has $\kappa_1(D, 0, \mu) > 0$ for any $D > 0$ in light of Theorem 3.1 in [41]. Thus, $\lambda_1(D, q, \mu)$ and $\kappa_1(D, q, \mu)$ are positive for any $D > 0$ and $q > 0$.

Let $S = e^{qx/D}\hat{S}$ in systems (2.1) and (2.2). Then

$$\begin{cases} D\hat{S}_{xx} + q\hat{S}_x + a(x)e^{-\frac{q}{D}x} - \mu(x)\hat{S} = 0, & 0 < x < L, \\ \hat{S}_x(0) = 0, \ D\hat{S}_x(L) + \nu q\hat{S}(L) = 0, \end{cases} \tag{2.7}$$

for $\nu \in (0, \infty)$, and

$$\begin{cases} D\hat{S}_{xx} + q\hat{S}_x + a(x)e^{-\frac{q}{D}x} - \mu(x)\hat{S} = 0, & 0 < x < L, \\ \hat{S}_x(0) = 0, \ \hat{S}(L) = 0. \end{cases} \tag{2.8}$$

Choose

$$\hat{S}_1 := \frac{\max\{a(x)e^{-\frac{q}{D}x} :\ 0 \le x \le L\}}{\min\{\mu(x) :\ 0 \le x \le L\}}, \quad \hat{S}_2 := \delta_{\Psi_1}\Psi_1,$$

and

$$\hat{S}_3 := \hat{S}_1, \quad \hat{S}_4 := \delta_{\Phi_1}\Phi_1,$$

wherein $0 < \delta_{\Psi_1} < \lambda_1^{-1} \min\{a(x)e^{-\frac{q}{D}x}\Psi_1^{-1} : 0 \le x \le L\}$ and $0 < \delta_{\Phi_1} < \kappa_1^{-1} \min\{a(x)e^{-\frac{q}{D}x}\Phi_1^{-1} : 0 \le x < L\}$. Through the definition of sub- and super-solutions for elliptic systems, it is not difficult to verify that $\hat{S}_1$, $\hat{S}_2$ and $\hat{S}_3$, $\hat{S}_4$ are a pair of super- and sub-solution of (2.7) and (2.8), respectively. Hence, systems (2.7) and (2.8) admit at least one positive solution, respectively.

Furthermore, assume $\hat{S}^*$ and $\hat{S}^{**}$ are two positive solutions of (2.7). Then $U := \hat{S}^* - \hat{S}^{**}$ fulfills

$$\begin{cases} DU_{xx} + qU_x - \mu(x)U = 0, & 0 < x < L, \\ U_x(0) = 0,\ DU_x(L) + \nu qU(L) = 0. \end{cases}$$

Therefore, by Theorem 6.31 in [8], the above system admits a unique solution and so $U \equiv 0$ in $[0, L]$. We can similarly deal with the case of $\nu = \infty$. This completes the proof. □

Remark 2.1. Thanks to Proposition 2.1, system (1.2) if (F3) holds and system (1.3) have a unique DFE, denoted by $E_0 := (P(x), 0, A(x), 0)$, respectively. Here, $P(\cdot)$ is the unique positive solution of (2.1) and (2.2) in $(0, L)$ where $D = D_1$, $q = a_h$, $\mu(\cdot) = d_h(\cdot)$ and $\nu = \nu_1 > 0$, respectively, and $A(\cdot)$ is the unique positive solution of (2.1) and (2.2) in $(0, L)$ where $D = D_4$, $q = a_v$, $\mu(\cdot) = d_v(\cdot)$ and $\nu = \nu_2 > 0$, respectively. Moreover, similar to the arguments of [37, Lemma 2.1], one obtains that $P(\cdot)$ and $A(\cdot)$ are global attractive in $C([0, L], \mathbb{R})$.

For simplicity, let

$$\mathbf{u}_0(\cdot) := (S_h^0(\cdot), I_h^0(\cdot), S_v^0(\cdot), I_v^0(\cdot)),\ \ \mathbf{u}(\cdot, \cdot; \mathbf{u}_0) := (S_h(\cdot,\cdot), I_h(\cdot,\cdot), S_v(\cdot,\cdot), I_v(\cdot,\cdot)),$$

and $\|\cdot\| := \|\cdot\|_{L^\infty((0,L))}$, $k^+ := \max\{k(x) : 0 \le x \le L\}$, $k^- := \min\{k(x) : 0 \le x \le L\}$, here $k(x)$ denotes the coefficients of (1.2) (resp. (1.3)). Applying the standard parabolic system theory yields that if $\mathbf{u}_0(\cdot) \in C([0, L], \mathbb{R}_+^4)$, then system (1.2) (resp. (1.3)) has a unique nonnegative classical solution $\mathbf{u}(\cdot, t; \mathbf{u}_0) \in C^{2,1}([0, L] \times (0, T_a))$ where $0 < T_a \le \infty$ is the maximum existence time of the solution. Moreover, $\mathbf{u}(\cdot, t; \mathbf{u}_0)$ is positive in $(0, L) \times (0, T_a)$ by (1.4). In what follows, we explore the global existence and ultimate boundedness.

Theorem 2.1. *For any $\mathbf{u}_0 \in C([0, L], \mathbb{R}_+^4)$ satisfying (1.4), system (1.2) (resp. (1.3)) possesses a unique nonnegative solution $\mathbf{u}(\cdot, t; \mathbf{u}_0)$ on $(0, L) \times (0, \infty)$. Furthermore, if $\nu_i \ge 1$, $i = 1, 2$, then $\mathbf{u}(\cdot, t; \mathbf{u}_0)$ is ultimately bounded, i.e., there exists a constant $C_1 > 0$ independent of initial data such that*

$$\limsup_{t\to\infty} [\|S_h(\cdot, t)\| + \|I_h(\cdot, t)\| + \|S_v(\cdot, t)\| + \|I_v(\cdot, t)\|] \le C_1. \tag{2.9}$$

Moreover, the solution semiflow $\Pi(t)\mathbf{u}_0 := \mathbf{u}(\cdot, t; \mathbf{u}_0)$ *has a global compact attractor.*

Proof. By adding the first two equations of (1.2) and integrating it over $(0, L)$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L (S_h + I_h)\mathrm{d}x &\le -\nu_1 a_h S_h(L,t) - \nu_1 q_h I_h(L,t) + \bar{H} - d_h^- \int_0^L (S_h + I_h)\mathrm{d}x \\ &\le \bar{H} - d_h^- \int_0^L (S_h + I_h)\mathrm{d}x, \quad \bar{H} := \int_0^L H(x)\mathrm{d}x.\end{aligned}$$

Utilizing the Gronwall's inequality yields

$$\int_0^L (S_h + I_h)\mathrm{d}x \le e^{-d_h^- t}\int_0^L [S_h^0(x) + I_h^0(x)]\mathrm{d}x + \frac{\bar{H}}{d_h^-}(1 - e^{-d_h^- t}).$$

In a similar fashion,

$$\int_0^L (S_v + I_v)\mathrm{d}x \le e^{-d_v^- t}\int_0^L [S_v^0(x) + I_v^0(x)]\mathrm{d}x + \frac{\bar{V}}{d_v^-}(1 - e^{-d_v^- t}), \quad \bar{V} := \int_0^L V(x)\mathrm{d}x.$$

In view of [13, Theorem 1], there is a constant $C_2 > 0$, depending on $I_h^0(\cdot)$ and $I_v^0(\cdot)$, such that $\|I_h(\cdot,t)\| + \|I_v(\cdot,t)\| \le C_2$, $t \in [0, T_a)$. Note that there exist two constants $C_3 = \|P(\cdot)\|$ and $C_4 = \|A(\cdot)\|$ such that $\|S_h(\cdot,t)\| \le C_3$ and $\|S_v(\cdot,t)\| \le C_4$, $t \in [0, T_a)$, from Remark 2.1. Hence, the solution of (1.2) exists globally on $[0, L] \times [0, \infty)$.

To achieve (2.9), we use the well-known induction method to show the following Claim.

Claim. *For any positive integer* m*, there exists a constant* $C_5 := C_5(m) > 0$*, such that*

$$\limsup_{t\to\infty}[\|S_h(\cdot,t)\|_{L^m((0,L))} + \|I_h(\cdot,t)\|_{L^m((0,L))} + \|S_v(\cdot,t)\|_{L^m((0,L))} + \|I_v(\cdot,t)\|_{L^m((0,L))}] \le C_5. \tag{2.10}$$

From the above arguments, (2.10) is true for $m = 1$. Suppose (2.10) holds for $m - 1$, i.e.,

$$\begin{aligned}&\limsup_{t\to\infty}[\|S_h(\cdot,t)\|_{L^{m-1}((0,L))} + \|I_h(\cdot,t)\|_{L^{m-1}((0,L))} \\ &\qquad + \|S_v(\cdot,t)\|_{L^{m-1}((0,L))} + \|I_v(\cdot,t)\|_{L^{m-1}((0,L))}] \le C_6,\end{aligned}$$

for some $C_6 := C_6(m) > 0$.

Through multiplying the first equation of (1.2) by S_h^{m-1} and then performing an integral in $(0, L)$, one has

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L S_h^m \mathrm{d}x = D_1\int_0^L S_h^{m-1}S_{hxx}\mathrm{d}x - a_h\int_0^L S_h^{m-1}S_{hx}\mathrm{d}x + \int_0^L S_h^{m-1}[H(x)-d_h(x)S_h]\mathrm{d}x - \int_0^L S_h^{m-1}g_1(x,S_h,I_v)\mathrm{d}x.$$

Since

$$\int_0^L S_h^{m-1}S_{hxx}\mathrm{d}x = S_h^{m-1}(L,t)S_{hx}(L,t) - S_h^{m-1}(0,t)S_{hx}(0,t) - (m-1)\int_0^L S_h^{m-2}S_{hx}^2\mathrm{d}x,$$

and

$$\int_0^L S_h^{m-1}S_{hx}\mathrm{d}x = S_h^m(L,t) - S_h^m(0,t) - (m-1)\int_0^L S_h^{m-1}S_{hx}\mathrm{d}x,$$

it follows that

$$\begin{aligned}&\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L S_h^m \mathrm{d}x + (m-1)D_1\int_0^L S_h^{m-2}S_{hx}^2\mathrm{d}x + a_h\left(1-\frac{1}{m}\right)S_h^m(0,t)\\ =&a_h\left(1-\nu_1-\frac{1}{m}\right)S_h^m(L,t) + \int_0^L S_h^{m-1}[H(x)-d_h(x)S_h]\mathrm{d}x - \int_0^L S_h^{m-1}g_1(x,S_h,I_v)\mathrm{d}x.\end{aligned}$$

Recalling that $\nu_1 \geq 1$ and $m > 1$, we have

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L S_h^m \mathrm{d}x \leq \int_0^L S_h^{m-1}[H(x)-d_h(x)S_h]\mathrm{d}x \leq H^+\int_0^L S_h^{m-1}\mathrm{d}x - d_h^-\int_0^L S_h^m\mathrm{d}x. \tag{2.11}$$

Similarly, multiplying the second, third and fourth equations of (1.2) by I_h^{m-1}, S_v^{m-1} and I_v^{m-1}, respectively, and then integrating it over $(0, L)$ to give

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L I_h^m \mathrm{d}x \leq \int_0^L I_h^{m-1}g_1(x,S_h,I_v)\mathrm{d}x - d_h^-\int_0^L I_h^m\mathrm{d}x, \tag{2.12}$$

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L S_v^m \mathrm{d}x \leq \int_0^L S_v^{m-1}[V(x)-d_v(x)S_h]\mathrm{d}x \leq V^+\int_0^L S_v^{m-1}\mathrm{d}x - d_v^-\int_0^L S_v^m\mathrm{d}x, \tag{2.13}$$

and

$$\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L I_v^m \mathrm{d}x \le \int_0^L I_v^{m-1} g_2(x,S_v,I_h)\mathrm{d}x - d_v^- \int_0^L I_v^m \mathrm{d}x. \tag{2.14}$$

Denote $\mathcal{M} := S_h^m + I_h^m + S_v^m + I_v^m$. By adding the inequalities (2.11)-(2.14), one obtains

$$\begin{aligned}\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \mathcal{M}\mathrm{d}x \le & H^+\int_0^L S_h^{m-1}\mathrm{d}x - d_h^-\int_0^L S_h^m\mathrm{d}x + \int_0^L I_h^{m-1}g_1(x,S_h,I_v)\mathrm{d}x - d_h^-\int_0^L I_h^m\mathrm{d}x \\ & + V^+\int_0^L S_v^{m-1}\mathrm{d}x - d_v^-\int_0^L S_v^m\mathrm{d}x + \int_0^L I_v^{m-1}g_2(x,S_v,I_h)\mathrm{d}x - d_v^-\int_0^L I_v^m\mathrm{d}x.\end{aligned}$$

Thus, by the hypothesis (F1) and above discussions, we have

$$g_1(x,S_h,I_v) \le \max\{\partial_{I_v} g_1(x,C_3,0) : 0 \le x \le L\} I_v := g_{1I_v}^+ I_v,$$

and

$$g_2(x,S_v,I_h) \le \max\{\partial_{I_h} g_2(x,C_4,0) : 0 \le x \le L\} I_v := g_{2I_h}^+ I_h.$$

Accordingly, it follows from the Young's inequality that

$$\int_0^L I_h^{m-1} I_v \mathrm{d}x \le \varepsilon_1 \int_0^L I_h^m \mathrm{d}x + C_{\varepsilon_1}(m)\int_0^L I_v^m \mathrm{d}x \text{ and}$$

$$\int_0^L I_h I_v^{m-1} \mathrm{d}x \le \varepsilon_2 \int_0^L I_h^m \mathrm{d}x + C_{\varepsilon_2}(m)\int_0^L I_v^m \mathrm{d}x,$$

for any $\varepsilon_i > 0$ and some positive constant $C_{\varepsilon_i}(m)$, $i = 1, 2$. Then

$$\begin{aligned}\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \mathcal{M}\mathrm{d}x \le & H^+\int_0^L S_h^{m-1}\mathrm{d}x + V^+\int_0^L S_v^{m-1}\mathrm{d}x - d_h^-\int_0^L (S_h^m + I_h^m)\mathrm{d}x - d_v^-\int_0^L (S_v^m + I_v^m)\mathrm{d}x \\ & + g_{1I_v}^+\int_0^L I_h^{m-1} I_v\mathrm{d}x + g_{2I_h}^+\int_0^L I_v^{m-1} I_h\mathrm{d}x \\ \le & H^+\int_0^L S_h^{m-1}\mathrm{d}x + V^+\int_0^L S_v^{m-1}\mathrm{d}x - d_h^-\int_0^L S_h^m\mathrm{d}x - d_v^-\int_0^L S_v^m\mathrm{d}x \\ & +(g_{1I_v}^+\varepsilon_1 + g_{2I_h}^+\varepsilon_2 - d_h^-)\int_0^L I_h^m\mathrm{d}x + [g_{1I_v}^+ C_{\varepsilon_1}(m) + g_{2I_h}^+ C_{\varepsilon_2}(m) - d_v^-]\int_0^L I_v^m\mathrm{d}x.\end{aligned}$$

Choosing suitable ε_i, $i = 1, 2$, such that $g^+_{1I_v}\varepsilon_1 + g^+_{2I_h}\varepsilon_2 - d^-_h \le -\frac{d^-_h}{2}$ and $g^+_{1I_v}C_{\varepsilon_1}(m) + g^+_{2I_h}C_{\varepsilon_2}(m) - d^-_v \le -\frac{d^-_v}{2}$. Therefore,

$$
\begin{aligned}
\frac{1}{m}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^L \mathcal{M}\mathrm{d}x \le & H^+\int_0^L S_h^{m-1}\mathrm{d}x + V^+\int_0^L S_v^{m-1}\mathrm{d}x - d_h^-\int_0^L S_h^m\mathrm{d}x - d_v^-\int_0^L S_v^m\mathrm{d}x \\
& - \frac{d_h^-}{2}\int_0^L I_h^m\mathrm{d}x - \frac{d_v^-}{2}\int_0^L I_v^m\mathrm{d}x \\
\le & H^+\int_0^L S_h^{m-1}\mathrm{d}x + V^+\int_0^L S_v^{m-1}\mathrm{d}x - \frac{\eta}{2}\int_0^L \mathcal{M}\mathrm{d}x,
\end{aligned}
$$

where $\eta := \min\{d_h^-, d_v^-\}$. Combining the assumption for $m-1$ and utilizing the Gronwall's inequality imply that (2.10) is valid for any positive integer m.

Consequently, by using the assertions in [13, Theorem 1], (2.9) holds which indicates that the solution of (1.2) with $\nu_i \in [1, \infty)$ is ultimately bounded. In a similar fashion, we can prove the global existence and ultimate boundedness of (1.3). Since system (1.2) (resp. (1.3)) is point dissipative and the semiflow $\Pi(t)$ is compact, it follows that $\Pi(t)$ admits a compact global attractor with the help of [9, Theorem 3.4.8]. This ends the proof. □

Remark 2.2. In fact, for some specific incidence functions, such as $g_i(\cdot, S, I) = \frac{k_i(\cdot)SI}{1+\rho_i(\cdot)I}$, $k_i(\cdot)$, $\rho_i(\cdot) > 0$, $i = 1, 2$, the ultimate boundedness of solutions when $\nu_i \in (0, \infty]$ can be directly obtained based on the classical comparison principle.

3. Threshold dynamics

In this section, we investigate the global dynamics of models (1.2) and (1.3).

3.1. Basic reproduction ratio

Linearizing systems (1.2) and (1.3) at E_0, respectively, to get

$$
\begin{cases}
\tilde{I}_{ht} = D_2\tilde{I}_{hxx} - q_h\tilde{I}_{hx} + k_1(x, P)\tilde{I}_v - \delta_h(x)\tilde{I}_h, & 0 < x < L,\ t > 0, \\
\tilde{I}_{vt} = D_5\tilde{I}_{vxx} - q_v\tilde{I}_{vx} + k_2(x, A)\tilde{I}_h - d_v(x)\tilde{I}_v, & 0 < x < L,\ t > 0, \\
D_2\tilde{I}_{hx}(0,t) - q_h\tilde{I}_h(0,t) = D_5\tilde{I}_{vx}(0,t) - q_v\tilde{I}_v(0,t) = 0, & t > 0, \\
D_2\tilde{I}_{hx}(L,t) - q_h\tilde{I}_h(L,t) = -\nu_1 q_h\tilde{I}_h(L,t), & t > 0, \\
D_5\tilde{I}_{vx}(L,t) - q_v\tilde{I}_v(L,t) = -\nu_2 q_v\tilde{I}_v(L,t), & t > 0,
\end{cases}
\tag{3.1}
$$

and

$$\begin{cases} \tilde{I}_{ht} = D_2\tilde{I}_{hxx} - q_h\tilde{I}_{hx} + k_1(x,P)\tilde{I}_v - \delta_h(x)\tilde{I}_h, & 0<x<L,\ t>0, \\ \tilde{I}_{vt} = D_5\tilde{I}_{vxx} - q_v\tilde{I}_{vx} + k_2(x,A)\tilde{I}_h - d_v(x)\tilde{I}_v, & 0<x<L,\ t>0, \\ D_2\tilde{I}_{hx}(0,t) - q_h\tilde{I}_h(0,t) = D_5\tilde{I}_{vx}(0,t) - q_v\tilde{I}_v(0,t) = 0, & t>0, \\ \tilde{I}_h(L,t) = \tilde{I}_v(L,t) = 0, & t>0, \end{cases} \tag{3.2}$$

wherein $k_1(\cdot,P) := \partial_{I_v} g_1(\cdot,P,0)$ and $k_2(\cdot,A) := \partial_{I_h} g_2(\cdot,A,0)$. Define the two operators $\mathcal{F}$, $\mathcal{B}$: $C([0,L],\mathbb{R}^2) \to C([0,L],\mathbb{R}^2)$ by

$$\mathcal{F}(\cdot) = \begin{pmatrix} 0 & k_1(\cdot,P) \\ k_2(\cdot,A) & 0 \end{pmatrix}, \quad -\mathcal{B}(\cdot) = \begin{pmatrix} D_2\partial_x^2 - q_h\partial_x - \delta_h(\cdot) & 0 \\ 0 & D_5\partial_x^2 - q_v\partial_x - d_v(\cdot) \end{pmatrix},$$

where ∂_x and ∂_x^2 are the first and second partial derivatives w.r.t x, respectively. Let

$$\mathfrak{L}[\upsilon](x) := \int_0^\infty \mathcal{F}(x)\tilde{T}(t)\upsilon(x)\mathrm{d}t,$$

here $\upsilon(x)$ is assumed to be the initial density distribution of infected hosts and vectors at location x, and $\tilde{T}(t)$ be the semigroup generated by $\mathrm{d}u/\mathrm{d}t = -\mathcal{B}u$ subject to the boundary conditions of (3.1) (resp. (3.2)). In light of [28], the basic reproduction ratio for system (1.2) (resp. (1.3)) is defined by the spectral radius of $\mathfrak{L}$, i.e.,

$$\mathcal{R}_0(D_2,D_5,q_h,q_v) = r(\mathfrak{L}) = \sup\{|\varrho|, \varrho \in \sigma(\mathfrak{L})\},$$

where $\sigma(\mathfrak{L})$ represents the spectral set of $\mathfrak{L}$. For convenience, denote $\mathcal{R}_0 := \mathcal{R}_0(D_2,D_5,q_h,q_v)$.

Consider two elliptic eigenvalue problems

$$\begin{cases} -D_2\varpi_{2xx} + q_h\varpi_{2x} + \delta_h(x)\varpi_2 = \varrho k_1(x,P)\varpi_5, & 0<x<L, \\ -D_5\varpi_{5xx} + q_v\varpi_{5x} + d_v(x)\varpi_5 = \varrho k_2(x,A)\varpi_2, & 0<x<L, \\ -D_2\varpi_{2x}(0) + q_h\varpi_2(0) = -D_5\varpi_{5x}(0) + q_v\varpi_5(0) = 0, & \\ -D_2\varpi_{2x}(L) + q_h\varpi_2(L) = \nu_1 q_h\varpi_2(L), & \\ -D_5\varpi_{5x}(L) + q_v\varpi_5(L) = \nu_2 q_v\varpi_5(L), & \end{cases} \tag{3.3}$$

and

$$\begin{cases} -D_2\varpi_{2xx} + q_h\varpi_{2x} + \delta_h(x)\varpi_2 = \varrho k_1(x,P)\varpi_5, & 0<x<L, \\ -D_5\varpi_{5xx} + q_v\varpi_{5x} + d_v(x)\varpi_5 = \varrho k_2(x,A)\varpi_2, & 0<x<L, \\ -D_2\varpi_{2x}(0) + q_h\varpi_2(0) = -D_5\varpi_{5x}(0) + q_v\varpi_5(0) = 0, & \\ \varpi_2(L) = \varpi_5(L) = 0. & \end{cases} \tag{3.4}$$

Lemma 3.1. *Suppose (F1)-(F3) hold. Let $\varrho_0 := \varrho_0(D_2,D_5,q_h,q_v)$ be the positive eigenvalue of (3.3) (resp. (3.4)) with positive eigenfunction. Then ϱ_0 is unique and $\mathcal{R}_0 = 1/\varrho_0$. Furthermore, if $k_1(\cdot,P) \equiv k_2(\cdot,A)$ in $(0,L)$, then $\mathcal{R}_0$ of (1.2) is given by*

$$\mathcal{R}_0 = \sup_{\substack{\varphi_2,\varphi_5\in H^1((0,L))\\ \varphi_2\neq 0,\ \varphi_5\neq 0}} \left\{\sqrt{\Theta_1(\varphi_2,\varphi_5)\Theta_2(\varphi_2,\varphi_5)}\right\}, \tag{3.5}$$

and $\mathcal{R}_0$ *of* (1.3) *is given by*

$$\mathcal{R}_0 = \sup_{\substack{\varphi_2,\varphi_5\in H^1((0,L))\\ \varphi_2\neq 0,\ \varphi_5\neq 0}} \left\{\sqrt{\Theta_3(\varphi_2,\varphi_5)\Theta_4(\varphi_2,\varphi_5)}\right\}, \tag{3.6}$$

where

$$\Theta_1(\varphi_2,\varphi_5) := \frac{\int_0^L k_1(x,P)e^{\frac{q_h}{D_2}x}\varphi_2\varphi_5\mathrm{d}x}{\nu_1 q_h e^{\frac{q_h}{D_2}L}\varphi_2^2(L) + D_2\int_0^L e^{\frac{q_h}{D_2}x}\varphi_{2x}^2\mathrm{d}x + \int_0^L \delta_h(x)e^{\frac{q_h}{D_2}x}\varphi_2^2\mathrm{d}x},$$

$$\Theta_2(\varphi_2,\varphi_5) := \frac{\int_0^L k_2(x,A)e^{\frac{q_v}{D_5}x}\varphi_2\varphi_5\mathrm{d}x}{\nu_2 q_v e^{\frac{q_v}{D_5}L}\varphi_5^2(L) + D_5\int_0^L e^{\frac{q_v}{D_5}x}\varphi_{5x}^2\mathrm{d}x + \int_0^L d_v(x)e^{\frac{q_v}{D_5}x}\varphi_5^2\mathrm{d}x},$$

and

$$\Theta_3(\varphi_2,\varphi_5) := \frac{\int_0^L k_1(x,P)e^{\frac{q_h}{D_2}x}\varphi_2\varphi_5\mathrm{d}x}{D_2\int_0^L e^{\frac{q_h}{D_2}x}\varphi_{2x}^2\mathrm{d}x + \int_0^L \delta_h(x)e^{\frac{q_h}{D_2}x}\varphi_2^2\mathrm{d}x},$$

$$\Theta_4(\varphi_2,\varphi_5) := \frac{\int_0^L k_2(x,A)e^{\frac{q_v}{D_5}x}\varphi_2\varphi_5\mathrm{d}x}{D_5\int_0^L e^{\frac{q_v}{D_5}x}\varphi_{5x}^2\mathrm{d}x + \int_0^L d_v(x)e^{\frac{q_v}{D_5}x}\varphi_5^2\mathrm{d}x}.$$

Proof. Similar to the arguments of [28, Lemma 2.1], [39, Lemma 2] and [31, Theorem 3.2 and Remark 3.1], one obtains that ϱ_0 is unique and $\mathcal{R}_0 = 1/\varrho_0$.

Let (ϖ_2,ϖ_5) be the positive eigenfunction corresponding to ϱ_0, and set $(\varpi_2,\varpi_5) = e^{\alpha x}(\varphi_2,\varphi_5)$ in (3.3). Through a simple calculation, one gets

$$\begin{cases} -D_2\varphi_{2xx} - q_h\varphi_{2x} + \delta_h(x)\varphi_2 = \varrho_0 k_1(x,P)\varphi_5, & 0<x<L,\\ -D_5\varphi_{5xx} - q_v\varphi_{5x} + d_v(x)\varphi_5 = \varrho_0 k_2(x,A)\varphi_2, & 0<x<L,\\ \varphi_{2x}(0) = \varphi_{5x}(0) = 0, & \\ D_2\varphi_{2x}(L) + \nu_1 q_h\varphi_2(L) = D_5\varphi_{5x}(L) + \nu_2 q_v\varphi_5(L) = 0. & \end{cases} \tag{3.7}$$

Multiplying the two equations of (3.7) by $e^{q_h x/D_2}$ and $e^{q_v x/D_5}$, respectively, we have

$$\begin{cases} -D_2(e^{\frac{q_h}{D_2}x}\varphi_{2x})_x + \delta_h(x)e^{\frac{q_h}{D_2}x}\varphi_2 = \dfrac{1}{\mathcal{R}_0}k_1(x,P)e^{\frac{q_h}{D_2}x}\varphi_5, & 0<x<L, \\ -D_5(e^{\frac{q_v}{D_5}x}\varphi_{5x})_x + d_v(x)e^{\frac{q_v}{D_5}x}\varphi_5 = \dfrac{1}{\mathcal{R}_0}k_2(x,A)e^{\frac{q_v}{D_5}x}\varphi_2, & 0<x<L, \\ \varphi_{2x}(0)=\varphi_{5x}(0)=0, \\ D_2\varphi_{2x}(L)+\nu_1 q_h\varphi_2(L) = D_5\varphi_{5x}(L)+\nu_2 q_v\varphi_5(L)=0. \end{cases}$$

Next, multiplying the two equations of above system by φ_2 and φ_5 respectively and then integrating by parts over $(0, L)$ to give

$$\begin{cases} \nu_1 q_h e^{\frac{q_h}{D_2}L}\varphi_2^2(L) + D_2\displaystyle\int_0^L e^{\frac{q_h}{D_2}x}\varphi_{2x}^2\mathrm{d}x + \int_0^L \delta_h(x)e^{\frac{q_h}{D_2}x}\varphi_2^2\mathrm{d}x = \frac{1}{\mathcal{R}_0}\int_0^L k_1(x,P)e^{\frac{q_h}{D_2}x}\varphi_2\varphi_5\mathrm{d}x, \\ \nu_2 q_v e^{\frac{q_v}{D_5}L}\varphi_5^2(L) + D_5\displaystyle\int_0^L e^{\frac{q_v}{D_5}x}\varphi_{5x}^2\mathrm{d}x + \int_0^L d_v(x)e^{\frac{q_v}{D_5}x}\varphi_5^2\mathrm{d}x = \frac{1}{\mathcal{R}_0}\int_0^L k_2(x,A)e^{\frac{q_v}{D_5}x}\varphi_2\varphi_5\mathrm{d}x. \end{cases}$$

As a result, by the variational methods of [32, Corollary of Theorem 2.3], the formula (3.5) is derived by multiplying the above two equalities and the definition of $\mathcal{R}_0$. Similarly, we can prove the formula (3.6). This completes the proof. □

Furthermore, we consider two elliptic eigenvalue problems as follows

$$\begin{cases} D_2u_{2xx} - q_hu_{2x} + k_1(x,P)u_5 - \delta_h(x)u_2 + \varrho u_2 = 0, & 0<x<L, \\ D_5u_{5xx} - q_vu_{5x} + k_2(x,A)u_2 - d_v(x)u_5 + \varrho u_5 = 0, & 0<x<L, \\ D_2u_{2x}(0) - q_hu_2(0) = D_5u_{5x}(0) - q_vu_5(0) = 0, & 0<x<L, \\ D_2u_{2x}(L) - q_hu_2(L) = -\nu_1 q_h u_2(L), \\ D_5u_{5x}(L) - q_vu_5(L) = -\nu_2 q_v u_5(L), \end{cases} \tag{3.8}$$

and

$$\begin{cases} D_2u_{2xx} - q_hu_{2x} + k_1(x,P)u_5 - \delta_h(x)u_2 + \varrho u_2 = 0, & 0<x<L, \\ D_5u_{5xx} - q_vu_{5x} + k_2(x,A)u_2 - d_v(x)u_5 + \varrho u_5 = 0, & 0<x<L, \\ D_2u_{2x}(0) - q_hu_2(0) = D_5u_{5x}(0) - q_vu_5(0) = 0, & 0<x<L, \\ u_2(L) = u_5(L) = 0. \end{cases} \tag{3.9}$$

Thanks to the transformation $(u_2, u_5) = e^{\alpha x}(w_2, w_5)$, systems (3.8) and (3.9) are rewritten as

$$\begin{cases} D_2 w_{2xx} + q_h w_{2x} + k_1(x,P) w_5 - \delta_h(x) w_2 + \varrho w_2 = 0, & 0 < x < L, \\ D_5 w_{5xx} + q_v w_{5x} + k_2(x,A) w_2 - d_v(x) w_5 + \varrho w_5 = 0, & 0 < x < L, \\ w_{2x}(0) = w_{5x}(0) = 0, \\ D_2 w_{2x}(L) + \nu_1 q_h w_2(L) = D_5 w_{5x}(L) + \nu_2 q_v w_5(L) = 0, \end{cases} \tag{3.10}$$

and

$$\begin{cases} D_2 w_{2xx} + q_h w_{2x} + k_1(x,P) w_5 - \delta_h(x) w_2 + \varrho w_2 = 0, & 0 < x < L, \\ D_5 w_{5xx} + q_v w_{5x} + k_2(x,A) w_2 - d_v(x) w_5 + \varrho w_5 = 0, & 0 < x < L, \\ w_{2x}(0) = w_{5x}(0) = 0,\ w_2(L) = w_5(L) = 0. \end{cases} \tag{3.11}$$

The Krein-Rutman theorem [11] illustrates that system (3.10) (resp. (3.11)) possesses a unique principal eigenvalue $\varrho_1 := \varrho_1(D_2, D_5, q_h, q_v)$. There are the following results with regard to the relationship between $\mathcal{R}_0$ and ϱ_1, and the proof resembles the Theorem 3.1 in [31], so is omitted.

Lemma 3.2. *Suppose (F1)-(F3) hold and $D_j > 0,\ q_b > 0,\ j = 2,5,\ b \in \{h,v\}$. Then $1 - \mathcal{R}_0$ share the same sign as ϱ_1, i.e., $sign(1-\mathcal{R}_0) = sign(\varrho_1)$.*

3.2. Stability of disease-free equilibrium

This subsection is devoted to discuss the global stability of DFE for (1.2) (resp. (1.3)), and fix $\nu_i \geq 1,\ i = 1,2$.

Lemma 3.3. *Let $\mathbf{u} = (S_h, I_h, S_v, I_v)$ be the solution of (1.2) (resp. (1.3)) satisfying $\mathbf{u}_0 \in C([0,L], \mathbb{R}^4_+)$.*

(i) *If there exist some $\tilde{t}_0 \geq 0$, such that $I_h(\cdot,\tilde{t}_0) \not\equiv 0$ and $I_v(\cdot,\tilde{t}_0) \not\equiv 0$ in $(0,L)$, then $I_h(x,t) > 0$ and $I_v(x,t) > 0$, for any $x \in (0,L)$ and $t > \tilde{t}_0$;*
(ii) *For any $\mathbf{u}_0 \in C([0,L], \mathbb{R}^4_+)$, then $S_h(x,t) > 0,\ S_v(x,t) > 0$ and*

$$\liminf_{t\to\infty} S_h(x,t) \geq \frac{H^-}{B_1 + d_h^+}, \quad \liminf_{t\to\infty} S_v(x,t) \geq \frac{V^-}{B_2 + d_v^+}, \quad \text{uniformly for } x \in (0,L),$$

where $B_1 := \max\{\partial_{S_h} g_1(x, S_h, C_1) : 0 \leq x \leq L, 0 \leq S_h \leq C_1\}$ and $B_2 := \max\{\partial_{S_v} g_2(x, S_v, C_1) : 0 \leq x \leq L, 0 \leq S_v \leq C_1\}$, here C_1 is given by Theorem 2.1.

Proof. From [39, Lemma 4], one can easily prove (i). According to Theorem 2.1, there exists a point t^* large enough, such that $|I_h(x,t)| + |I_v(x,t)| \leq C_1$ for any $x \in [0,L]$ and $t > t^*$. By the first equation of (1.2), we have

$$\begin{cases} S_{ht} \geq D_1 S_{hxx} - a_h S_{hx} + H^- - (B_1 + d_h^+) S_h, & 0 < x < L,\ t > t^*, \\ D_1 S_{hx}(0,t) - a_h S_h(0,t) = 0, & t > t^*, \\ D_1 S_{hx}(L,t) - a_h S_h(L,t) = -\nu_1 a_h S_h(L,t), & t > t^*. \end{cases}$$

Then the comparison principle yields that

$$\liminf_{t\to\infty} S_h(x,t) \geq \frac{H^-}{B_1 + d_h^+} \quad \text{uniformly for } x \in (0, L).$$

In a similar fashion, one can deduce the estimate of S_v in system (1.2) and deal with the system (1.3). This ends the proof. $\square$

Lemma 3.4. *Suppose (F1)-(F3) hold. If $\mathcal{R}_0 \leq 1$, then E_0 of* (1.2) *(resp.* (1.3)*) is globally attractive, namely,*

$$\lim_{t\to\infty} (S_h(x,t), I_h(x,t), S_v(x,t), I_v(x,t)) = (P(x), 0, A(x), 0)$$

uniformly for $x \in [0, L]$.

Proof. We first cope with the model (1.2). By Theorem 2.1, there is a constant $C_7 > 0$ such that $\Pi(t)\mathbf{u}_0 \subset \Omega$, where $\Omega := \{\mathbf{u} \in C([0, L] \times [0, \infty), \mathbb{R}_+^4) | 0 < S_h, I_h, S_v, I_v \leq C_7\}$. To show that $(I_h(\cdot, t), I_v(\cdot, t)) \to (0, 0)$ as $t \to \infty$ uniformly in $[0, L]$ when $\mathcal{R}_0 \leq 1$ via applying the methods of [25, Theorem 5.1].

Define

$$c(t; \mathbf{u}) := \max \left\{ \max_{x\in[0,L]} \frac{I_h(x,t)}{e^{-\varrho_1 t} u_2(x)}, \max_{x\in[0,L]} \frac{I_v(x,t)}{e^{-\varrho_1 t} u_5(x)} \right\}, \quad t > 0,$$

where $\mathbf{u} = (S_h, I_h, S_v, I_v)$ and ϱ_1 is the principal eigenvalue of (3.8) with positive eigenfunction (u_2, u_5). Note that $(u_2, u_5) = e^{\alpha x}(w_2, w_5)$, here (w_2, w_5) is the solution of (3.10). Then $\varrho_1 \geq 0$ owing to Lemma 3.2. From Lemma 3.3, if there exists a $\tilde{t}_0 \geq 0$, such that $I_h(\cdot, \tilde{t}_0) \not\equiv 0$ and $I_v(\cdot, \tilde{t}_0) \not\equiv 0$, then $I_h(x,t) > 0$ and $I_v(x,t) > 0$, for any $t > \tilde{t}_0$, $x \in [0, L]$. Since $S_h(x,t) \leq P(x)$ and $S_v(x,t) \leq A(x)$, $t > 0$, $x \in [0, L]$, it follows from the hypothesis (F1) that

$$\begin{cases} I_{ht} \leq D_2 I_{hxx} - q_h I_{hx} + k_1(x, P) I_v - \delta_h(x) I_h, & 0 < x < L, \ t > 0, \\ I_{vt} \leq D_5 I_{vxx} - q_v I_{vx} + k_2(x, A) I_h - d_v(x) I_v, & 0 < x < L, \ t > 0. \end{cases}$$

Moreover, we can verify that $(c(\tilde{t}_1; \mathbf{u}) e^{-\varrho_1 t} u_2(x), c(\tilde{t}_1; \mathbf{u}) e^{-\varrho_1 t} u_5(x))$ is a positive solution of the linearized system (3.1) for any $\tilde{t}_1 \geq \tilde{t}_0$. Then, the strong maximum principle implies that

$$\begin{aligned} &I_h(x,t) < c(\tilde{t}_1; \mathbf{u}) e^{-\varrho_1 t} u_2(x), \ \ I_v(x,t) < c(\tilde{t}_1; \mathbf{u}) e^{-\varrho_1 t} u_5(x), \\ &\text{for any } t > \tilde{t}_1 \geq \tilde{t}_0, \ x \in (0, L), \end{aligned}$$

which indicates that

$$\frac{I_h(x,t)}{e^{-\varrho_1 t} u_2(x)} < c(\tilde{t}_1; \mathbf{u}), \quad \frac{I_v(x,t)}{e^{-\varrho_1 t} u_5(x)} < c(\tilde{t}_1; \mathbf{u}), \ \text{ for any } t > \tilde{t}_1 \geq \tilde{t}_0, \ x \in (0, L).$$

Then $c(t;\mathbf{u}) < c(\tilde{t}_1;\mathbf{u})$ for $t > \tilde{t}_1$, which gives that $c(t;\mathbf{u})$ is strictly decreasing in t. For $\varrho_1 > 0$ (i.e., $\mathcal{R}_0 < 1$), it is obvious that $(I_h(x,t), I_v(x,t)) \to (0,0)$ as $t \to \infty$.

To prove $(I_h(\cdot,t), I_v(\cdot,t)) \to (0,0)$ as $t \to \infty$ as $\varrho_1 = 0$ (i.e., $\mathcal{R}_0 = 1$). Combining $c(t;\mathbf{u}) \geq 0$ and the monotonicity of $c(t;\mathbf{u})$ w.r.t t, we get $\lim_{t\to\infty} c(t;\mathbf{u}) = c_*$ for some constant $c_* \geq 0$. Suppose $c_* > 0$. Then, there is a subsequence t_k meeting $t_k \to \infty$ when $k \to \infty$ such that

$$\mathbf{u}(\cdot, t+t_k) = (S_h(\cdot, t+t_k), I_h(\cdot, t+t_k), S_v(\cdot, t+t_k), I_v(\cdot, t+t_k)) \to \mathbf{u}^*(\cdot, t) \text{ as } k \to \infty,$$

wherein $\mathbf{u}^*(\cdot,t) := (\tilde{S}_h^*(\cdot,t), \tilde{I}_h^*(\cdot,t), \tilde{S}_v^*(\cdot,t), \tilde{I}_v^*(\cdot,t))$, and $\tilde{I}_h^*(\cdot,t)$ or $\tilde{I}_v^*(\cdot,t)$ is not identically zero, and $\tilde{S}_h^*(\cdot,t) \leq P(\cdot)$ and $\tilde{S}_v^*(\cdot,t) \leq A(\cdot)$ in $[0,L]$, for all $t > 0$. Through similar arguments, we can show that $c(t;\mathbf{u}^*)$ is strictly decreasing for all sufficiently large t. On the other side, $c(t;\mathbf{u}^*) = \lim_{k\to\infty} c(t+t_k;\mathbf{u}) = c_*$ which contradicts the monotonicity of $c(t;\mathbf{u}^*)$ and so $c_* = 0$. Hence, $I_h(\cdot,t) \to 0$ and $I_v(\cdot,t) \to 0$ as $t \to \infty$ in $(0,L)$ when $\mathcal{R}_0 = 1$.

In addition, resembling the proof in [39, Theorem 2], one has $(S_h(\cdot,t), S_v(\cdot,t)) \to (P(\cdot), A(\cdot))$ as $t \to \infty$ uniformly in $[0,L]$ with the aid of the internally chain transitive set theory [40] and Lemma 3.3. Thus, E_0 is globally attractive. Similar arguments can be used to substantiate the global attractivity of E_0 for (1.3). This completes the proof. □

As a result, the main conclusions of this part are as follows:

Theorem 3.1. *Suppose (F1)-(F3) hold. If $\mathcal{R}_0 < 1$, then E_0 of (1.2) (resp. (1.3)) is globally asymptotically stable (GAS).*

Proof. In view of [31, Theorem 3.1], E_0 is asymptotically stable if $\mathcal{R}_0 < 1$. The global stability follows Lemma 3.4. □

3.3. Uniform persistence

In this subsection, we discuss the uniform persistence of (1.2) (resp. (1.3)) when $\mathcal{R}_0 > 1$, and fix $\nu_i \geq 1$, $i = 1, 2$.

Lemma 3.5. *Suppose (F1)-(F3) hold. If $\mathcal{R}_0 > 1$, there exists a constant $\epsilon_0 > 0$, such that the solution of (1.2) (resp. (1.3)) fulfills*

$$\liminf_{t\to\infty} S_h(x,t) \geq \epsilon_0,\ \liminf_{t\to\infty} I_h(x,t) \geq \epsilon_0,\ \liminf_{t\to\infty} S_v(x,t) \geq \epsilon_0,\ \liminf_{t\to\infty} I_v(x,t) \geq \epsilon_0 \tag{3.12}$$

uniformly for $x \in (0,L)$.

Proof. For system (1.2), let

$$\begin{aligned}&(S_h(x,\cdot), I_h(x,\cdot), S_v(x,\cdot), I_v(x,\cdot)) \\ &\quad = \left(e^{\frac{a_h}{D_1}x}\tilde{S}_h(x,\cdot), e^{\frac{q_h}{D_2}x}\tilde{I}_h(x,\cdot), e^{\frac{a_v}{D_4}x}\tilde{S}_v(x,\cdot), e^{\frac{q_v}{D_5}x}\tilde{I}_v(x,\cdot)\right).\end{aligned}$$

Then $(\tilde{S}_h(x,\cdot), \tilde{I}_h(x,\cdot), \tilde{S}_v(x,\cdot), \tilde{I}_v(x,\cdot))$ satisfies

$$\begin{cases} \tilde{S}_{ht} = D_1\tilde{S}_{hxx} + a_h\tilde{S}_{hx} + H_1(x) - g_{11}(x,\tilde{S}_h,\tilde{I}_v) - d_h(x)\tilde{S}_h, & 0<x<L,\ t>0,\\ \tilde{I}_{ht} = D_2\tilde{I}_{hxx} + q_h\tilde{I}_{hx} + g_{12}(x,\tilde{S}_h,\tilde{I}_v) - \delta_h(x)\tilde{I}_h, & 0<x<L,\ t>0,\\ \tilde{S}_{vt} = D_4\tilde{S}_{vxx} + a_v\tilde{S}_{vx} + V_1(x) - g_{21}(x,\tilde{S}_v,\tilde{I}_h) - d_v(x)\tilde{S}_v, & 0<x<L,\ t>0,\\ \tilde{I}_{vt} = D_5\tilde{I}_{vxx} + q_v\tilde{I}_{vx} + g_{22}(x,\tilde{S}_v,\tilde{I}_h) - d_v(x)\tilde{I}_v, & 0<x<L,\ t>0,\\ \tilde{S}_{hx}(0,t) = \tilde{I}_{hx}(0,t) = \tilde{S}_{vx}(0,t) = \tilde{I}_{vx}(0,t) = 0, & t>0,\\ D_1\tilde{S}_{hx}(L,t) + \nu_1 a_h\tilde{S}_h(L,t) = D_2\tilde{I}_{hx}(L,t) + \nu_1 q_h\tilde{I}_h(L,t) = 0, & t>0,\\ D_4\tilde{S}_{vx}(L,t) + \nu_2 a_v\tilde{S}_v(L,t) = D_5\tilde{I}_{vx}(L,t) + \nu_2 q_v\tilde{I}_v(L,t) = 0, & t>0, \end{cases} \tag{3.13}$$

where $H_1(x) := e^{-a_hx/D_1}H(x)$, $V_1(x) := e^{-a_vx/D_4}V(x)$ and

$$g_{11}(x,\tilde{S}_h,\tilde{I}_v) := e^{-\frac{a_h}{D_1}x}g_1(x, e^{\frac{a_h}{D_1}x}\tilde{S}_h, e^{\frac{q_v}{D_5}x}\tilde{I}_v),\ \ g_{12}(x,\tilde{S}_h,\tilde{I}_v) := e^{-\frac{q_h}{D_2}x}g_1(x, e^{\frac{a_h}{D_1}x}\tilde{S}_h, e^{\frac{q_v}{D_5}x}\tilde{I}_v),$$

and

$$g_{21}(x,\tilde{S}_v,\tilde{I}_h) := e^{-\frac{a_v}{D_4}x}g_2(x, e^{\frac{a_v}{D_4}x}\tilde{S}_v, e^{\frac{q_h}{D_2}x}\tilde{I}_h),\ \ g_{22}(x,\tilde{S}_v,\tilde{I}_h) := e^{-\frac{q_v}{D_5}x}g_2(x, e^{\frac{a_v}{D_4}x}\tilde{S}_v, e^{\frac{q_h}{D_2}x}\tilde{I}_h).$$

From Remark 2.1, it follows that system (3.13) has a unique DEF $\tilde{E}_0 := (\tilde{P}(x), 0, \tilde{A}(x), 0)$, where $\tilde{P}(x) = e^{-a_hx/D_1}P(x)$ and $\tilde{A}(x) = e^{-a_vx/D_4}A(x)$. We first show there is a constant $\tilde{\epsilon}_0 > 0$ such that

$$\liminf_{t\to\infty}\tilde{S}_h(\cdot,t) \ge \tilde{\epsilon}_0,\ \liminf_{t\to\infty}\tilde{I}_h(\cdot,t) \ge \tilde{\epsilon}_0,\ \liminf_{t\to\infty}\tilde{S}_v(\cdot,t) \ge \tilde{\epsilon}_0,\ \liminf_{t\to\infty}\tilde{I}_v(\cdot,t) \ge \tilde{\epsilon}_0 \tag{3.14}$$

uniformly in $[0, L]$. Denote $\tilde{\mathbf{u}} := (\tilde{S}_h, \tilde{I}_h, \tilde{S}_v, \tilde{I}_v)$ and $\tilde{\mathbf{u}}_0 := (\tilde{S}_h^0, \tilde{I}_h^0, \tilde{S}_v^0, \tilde{I}_v^0)$. Then $(\tilde{S}_h^0, \tilde{I}_h^0, \tilde{S}_v^0, \tilde{I}_v^0) = (e^{-a_hx/D_1}S_h^0, e^{-q_hx/D_2}I_h^0, e^{-a_vx/D_4}S_v^0, e^{-q_vx/D_5}I_v^0)$. By Theorem 2.1, the solution of (3.13) lies in set $\mathbf{E}$, where $\mathbf{E} := \{\tilde{\mathbf{u}} \in C([0,L]\times[0,\infty), \mathbb{R}_+^4) | 0 \le \tilde{S}_h, \tilde{I}_h, \tilde{S}_v, \tilde{I}_v \le C_8\}$ for some $C_8 > 0$. Let

$$\Lambda_0 := \left\{\tilde{\mathbf{u}}_0 \in \mathbf{E} \,|\, \tilde{I}_h^0 \not\equiv 0 \text{ and } \tilde{I}_v^0 \not\equiv 0\right\},\ \ \partial\Lambda_0 := \left\{\tilde{\mathbf{u}}_0 \in \mathbf{E} \,|\, \tilde{I}_h^0 \equiv 0 \text{ or } \tilde{I}_v^0 \equiv 0\right\}.$$

It is not difficult to see that $\mathbf{E} = \Lambda_0 \cup \partial\Lambda_0$, where Λ_0 and $\partial\Lambda_0$ are open convex and closed sets of $\mathbf{E}$, respectively. Let $\tilde{\Pi}(t)\tilde{\mathbf{u}}_0 = \tilde{\mathbf{u}}$ be the unique solution of (3.13) satisfying $\tilde{\mathbf{u}}_0 \in \mathbf{E}$, $t > 0$. Thus, by using Theorem 2.1 and the strong maximum principle, $\tilde{\Pi}(t)$ has the global compact attractor and $\tilde{\Pi}(t)\Lambda_0 \subset \Lambda_0$. Denote $\mathbb{Z}_\partial$ represent the maximum positively invariant set of $\tilde{\Pi}(t)$ in $\partial\Lambda_0$, that is, $\mathbb{Z}_\partial = \{\tilde{\mathbf{u}}_0 \in \mathbf{E} \,|\, \tilde{\Pi}(t)\tilde{\mathbf{u}}_0 \in \partial\Lambda_0\}$. Then $\mathbb{Z}_\partial = \{\tilde{\mathbf{u}}_0 \in \mathbf{E} \,|\, \tilde{I}_h^0 = \tilde{I}_v^0 \equiv 0\}$. We let $\omega(\tilde{\mathbf{u}}_0)$ be the omega limit set of $\tilde{\mathbf{u}}_0$ in $\mathbf{E}$ and $\bar{\mathbb{Z}}_\partial := \bigcup_{\tilde{\mathbf{u}}_0\in\mathbb{Z}_\partial}\omega(\tilde{\mathbf{u}}_0)$. To end the proof, we prove the following claims.

Claim 1. $\bar{\mathbb{Z}}_\partial = \{\tilde{E}_0\}$.

In fact, for any $\tilde{\mathbf{u}}_0 \in \mathbb{Z}_\partial$, by the definition of $\mathbb{Z}_\partial$, one has $\tilde{I}_h(x,t) = \tilde{I}_v(x,t) = 0$ for any $x \in [0, L]$ and $t \geq 0$. Substituting it into (3.13) yields

$$\begin{cases} \tilde{S}_{ht} = D_1 \tilde{S}_{hxx} + a_h \tilde{S}_{hx} + H_1(x) - d_h(x)\tilde{S}_h, & 0 < x < L,\ t > 0, \\ \tilde{S}_{vt} = D_4 \tilde{S}_{vxx} + a_v \tilde{S}_{vx} + V_1(x) - d_v(x)\tilde{S}_v, & 0 < x < L,\ t > 0, \\ \tilde{S}_{hx}(0,t) = \tilde{S}_{vx}(0,t) = 0, & t > 0, \\ D_1 \tilde{S}_{hx}(L,t) + \nu_1 a_h \tilde{S}_h(L,t) = D_4 \tilde{S}_{vx}(L,t) + \nu_2 a_v \tilde{S}_v(L,t) = 0, & t > 0. \end{cases}$$

By employing Remark 2.1, $\tilde{P}(\cdot)$ and $\tilde{A}(\cdot)$ are global attractive, i.e., $\tilde{S}_h(\cdot,t) \to \tilde{P}(\cdot)$ and $\tilde{S}_v(\cdot,t) \to \tilde{A}(\cdot)$ as $t \to \infty$ uniformly in $(0, L)$ which implies Claim 1 holds and $\{\tilde{E}_0\}$ is an isolated and compact invariant set for $\tilde{\Pi}(t)$ restricted in $\bar{\mathbb{Z}}_\partial$.

Claim 2. *There is a constant $\epsilon_1 > 0$, which does not depend on initial values, such that*

$$\liminf_{t\to\infty} \|\tilde{\Pi}(t)\tilde{\mathbf{u}}_0 - (\tilde{P}(\cdot), 0, \tilde{A}(\cdot), 0)\| > \epsilon_1 \ \text{ uniformly in } [0, L].$$

Arguing by contradiction, for any $\hat{\epsilon}_1 > 0$, there is $\hat{\mathbf{u}}_0 = (\hat{S}_h^0, \hat{I}_h^0, \hat{S}_v^0, \hat{I}_v^0)$ such that

$$\liminf_{t\to\infty} \|\hat{\Pi}(t)\hat{\mathbf{u}}_0 - (\tilde{P}(\cdot), 0, \tilde{A}(\cdot), 0)\| \leq \hat{\epsilon}_1, \tag{3.15}$$

where $\hat{\Pi}(t)\hat{\mathbf{u}}_0 = (\hat{S}_h(\cdot,t), \hat{I}_h(\cdot,t), \hat{S}_v(\cdot,t), \hat{I}_v(\cdot,t))$. For any fixed $\epsilon_2 > 0$ small enough, let $\varrho_1^{\epsilon_2} = \varrho_1(D_2, D_5, q_h, q_v, \epsilon_2)$ be the principal eigenvalue of problem

$$\begin{cases} D_2 w_{2xx} + q_h w_{2x} + \hat{k}_1(x, \tilde{P} - \epsilon_2) w_5 - \delta_h(x) w_2 + \varrho_1^{\epsilon_2} w_2 = 0, & 0 < x < L, \\ D_5 w_{5xx} + q_v w_{5x} + \hat{k}_2(x, \tilde{A} - \epsilon_2) w_2 - d_v(x) w_5 + \varrho_1^{\epsilon_2} w_5 = 0, & 0 < x < L, \\ w_{2x}(0) = w_{5x}(0) = 0, \\ D_2 w_{2x}(L) + \nu_1 q_h w_2(L) = D_5 w_{5x}(L) + \nu_2 q_v w_5(L) = 0, \end{cases}$$

with positive eigenfunction (w_2, w_5), wherein

$$\hat{k}_1(x, \tilde{P} - \epsilon_2) = \partial_{\tilde{I}_v} g_1(x, e^{\frac{a_h}{D_1}x}(\tilde{P} - \epsilon_2), \epsilon_2) \ \text{ and } \ \hat{k}_2(x, \tilde{A} - \epsilon_2) = \partial_{\tilde{I}_h} g_2(x, e^{\frac{a_v}{D_4}x}(\tilde{A} - \epsilon_2), \epsilon_2).$$

Recalling that $\mathcal{R}_0 > 1$, we have $\varrho_1 < 0$. Since $\varrho_1^{\epsilon_2} \to \varrho_1$ as $\epsilon_2 \to 0$, it follows that there is a sufficiently small ϵ_2 such that $\varrho_1^{\epsilon_2} < 0$. Without loss of generality, choose $\hat{\epsilon}_1 = \epsilon_2$. From (3.15), there is a $\hat{t}_0 \geq 0$ such that $\hat{S}_h(\cdot,t) \geq \tilde{P}(\cdot) - \epsilon_2$, $\hat{S}_v(\cdot,t) \geq \tilde{A}(\cdot) - \epsilon_2$, $\hat{I}_h(\cdot,t) \leq \epsilon_2$ and $\hat{I}_v(\cdot,t) \leq \epsilon_2$, for any $x \in [0, L]$ and $t \geq \hat{t}_0$. Therefore,

$$g_{12}(x, \hat{S}_h, \hat{I}_v) \geq \partial_{\tilde{I}_v} g_1(x, e^{\frac{a_h}{D_1}x}(\tilde{P} - \epsilon_2), \epsilon_2)\hat{I}_v = \hat{k}_1(x, \tilde{P} - \epsilon_2)\hat{I}_v,$$

and

$$g_{22}(x,\hat{S}_v,\hat{I}_h) \geq \partial_{\tilde{I}_h} g_2(x, e^{\frac{a_v}{D_4}x}(\tilde{A}-\epsilon_2),\epsilon_2)\hat{I}_h = \hat{k}_2(x,\tilde{A}-\epsilon_2)\hat{I}_h.$$

By appealing to Theorem 2.1 and strong maximum principle, there is constant $\zeta_0>0$ such that $\hat{I}_h(x,\hat{t}_0) \geq \zeta_0 w_2(x)$ and $\hat{I}_v(x,\hat{t}_0) \geq \zeta_0 w_5(x)$ for any $x\in[0,L]$. It is straightforward that $(\hat{I}_h,\hat{I}_v)$ is a super-solution of the following system

$$\begin{cases} \bar{I}_{ht} = D_2\bar{I}_{hxx} + q_h\bar{I}_{hx} + \hat{k}_1(x,\tilde{P}-\epsilon_2)\bar{I}_v - \delta_h(x)\bar{I}_h, & 0<x<L,\ t>\hat{t}_0,\\ \bar{I}_{vt} = D_5\bar{I}_{vxx} + q_v\bar{I}_{vx} + \hat{k}_2(x,\tilde{A}-\epsilon_2)\bar{I}_h - d_v(x)\bar{I}_v, & 0<x<L,\ t>\hat{t}_0,\\ \bar{I}_{hx}(0,t) = \bar{I}_{vx}(0,t) = 0, & t>\hat{t}_0,\\ D_2\bar{I}_{hx}(L,t) + \nu_1 q_h\bar{I}_h(L,t) = D_5\bar{I}_{vx}(L,t) + \nu_2 q_v\bar{I}_v(L,t) = 0, & t>\hat{t}_0,\\ \bar{I}_h(x,\hat{t}_0) = \zeta_0 w_2(x),\ \bar{I}_v(x,\hat{t}_0) = \zeta_0 w_5(x), & 0<x<L. \end{cases} \tag{3.16}$$

Noting that $(\zeta_0 e^{-\varrho_1^{\epsilon_2}(t-\hat{t}_0)}w_2, \zeta_0 e^{-\varrho_1^{\epsilon_2}(t-\hat{t}_0)}w_5)$ is a solution of (3.16), so the comparison principle implies that

$$\hat{I}_h(x,t) \geq \zeta_0 e^{-\varrho_1^{\epsilon_2}(t-\hat{t}_0)}w_2(x),\ \ \hat{I}_v(x,t) \geq \zeta_0 e^{-\varrho_1^{\epsilon_2}(t-\hat{t}_0)}w_5(x),\ \ t\geq\hat{t}_0,\ x\in[0,L].$$

Then $\hat{I}_h(x,t)\to\infty$ and $\hat{I}_v(x,t)\to\infty$, $t\to\infty$, because of $\varrho_1^{\epsilon_2}<0$, which contradicts (3.15) and thereby deduces Claim 2. Hence, $\{\tilde{E}_0\}$ is an isolated invariant set for $\tilde{\Pi}(t)$ restricted in $\mathbf{E}$, and $W^S(\{\tilde{E}_0\})\cap\Lambda_0=\emptyset$, where $W^S(\{\tilde{E}_0\})$ represents the stable set of $\{\tilde{E}_0\}$ w.r.t $\tilde{\Pi}(t)$.

By Claims 1-2 and Theorem 1.3.1 in [40], $\tilde{\Pi}(t)$ is uniformly persistent for $(\mathbb{E},\partial\Lambda_0)$. Therefore, we complete the proof of (3.14), and so (3.12) is valid. Similarly, the conclusions of Lemma 3.5 hold for system (1.3). This finishes the proof. □

Theorem 3.2. *Suppose (F1)-(F3) hold. If $\mathcal{R}_0>1$, there exists a constant $\epsilon_3>0$, such that the solution of* (1.2) *(resp.* (1.3)*) fulfills*

$$\liminf_{t\to\infty} S_h(x,t)\geq\epsilon_3,\ \ \liminf_{t\to\infty} S_v(x,t)\geq\epsilon_3, \tag{3.17}$$

and

$$\liminf_{t\to\infty} I_h(x,t)\geq\frac{g_1^-(\epsilon_3,\epsilon_0)}{\delta_h^+},\ \ \liminf_{t\to\infty} I_v(x,t)\geq\frac{g_2^-(\epsilon_3,\epsilon_0)}{d_v^+} \tag{3.18}$$

uniformly for $x\in(0,L)$, where $g_i^-(\epsilon_3,\epsilon_0):=\min\{g_i(x,\epsilon_3,\epsilon_0):0\leq x\leq L\}$, $i=1,2$. Furthermore, system (1.2) *(resp.* (1.3)*) has at least one EE.*

Proof. For system (1.2), by Lemma 3.5, the inequality (3.17) is obvious and there exists a point $t_2 \geq 0$, such that $|I_h(x,t)| \geq \epsilon_0$ and $|I_v(x,t)| \geq \epsilon_0$ for any $x \in [0, L]$ and $t > t_2$. Thus, from the second and fourth equations of (1.2) and (F1), one gets

$$\begin{cases} I_{ht} \geq D_2 I_{hxx} - q_h I_{hx} + g_1(x, \epsilon_3, \epsilon_0) - \delta_h(x) I_h, & 0 < x < L,\ t > t_2, \\ I_{vt} \geq D_5 I_{vxx} - q_v I_{vx} + g_2(x, \epsilon_3, \epsilon_0) - d_v(x) I_v, & 0 < x < L,\ t > t_2, \\ D_2 I_{hx}(0,t) - q_h I_h(0,t) = D_5 I_{vx}(0,t) - q_v I_v(0,t) = 0, & t > t_2, \\ D_2 I_{hx}(L,t) - q_h I_h(L,t) = -\nu_1 q_h I_h(L,t), & t > t_2, \\ D_5 I_{vx}(L,t) - q_v I_v(L,t) = -\nu_2 q_v I_v(L,t), & t > t_2. \end{cases}$$

Then utilizing the comparison principle yields that (3.18) holds. Moreover, system (1.2) admits at least one endemic steady state when $\mathcal{R}_0 > 1$ thanks to [40, Theorem 1.3.7]. One can similarly deal with system (1.3). This completes the proof. □

4. Monotonicity and asymptotic profiles of basic reproduction ratio

The monotonicity of $\mathcal{R}_0$ for systems (1.2) and (1.3) w.r.t the advection rates q_h and q_v is as follows:

Proposition 4.1. *Suppose the conditions of Lemma 3.1 hold and $\nu_i \geq 1/2$, $i = 1, 2$. For any fixed $D_2 > 0$ and $D_5 > 0$, $\mathcal{R}_0$ is strictly monotonically decreasing w.r.t q_h and q_v, respectively.*

Proof. Similar to the arguments of [1, Proposition 2.20] and [10, Lemma 15.1], $\mathcal{R}_0$ and the eigenfunction (ϖ_2, ϖ_5) of (3.3) are differentiable w.r.t q_h and q_v, respectively. Since $(\varpi_2, \varpi_5) = e^{\alpha x}(\varphi_2, \varphi_5)$, it follows that (φ_2, φ_5) is differentiable w.r.t q_h and q_v, respectively. Differentiating system (3.7) w.r.t q_h to yield

$$\begin{cases} -D_2 \dot{\varphi}_{2xx} - \varphi_{2x} - q_h \dot{\varphi}_{2x} + \delta_h(x) \dot{\varphi}_2 = -\dfrac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2} k_1(x, P) \varphi_5 + \dfrac{1}{\mathcal{R}_0} k_1(x, P) \dot{\varphi}_5, & 0 < x < L, \\ -D_5 \dot{\varphi}_{5xx} - q_v \dot{\varphi}_{5x} + d_v(x) \dot{\varphi}_5 = -\dfrac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2} k_2(x, A) \varphi_2 + \dfrac{1}{\mathcal{R}_0} k_2(x, A) \dot{\varphi}_2, & 0 < x < L, \\ \dot{\varphi}_{2x}(0) = \dot{\varphi}_{5x}(0) = 0, & \\ D_2 \dot{\varphi}_{2x}(L) + \nu_1 \varphi_2(L) + \nu_1 q_h \dot{\varphi}_2(L) = D_5 \dot{\varphi}_{5x}(L) + \nu_2 q_v \dot{\varphi}_5(L) = 0, & \end{cases} \tag{4.1}$$

where $\dot{}$ denotes the derivative w.r.t q_h. By multiplying the first and second equations of (4.1) by $e^{q_h x/D_2} \varphi_2$ and $e^{q_v x/D_5} \varphi_5$, and integrating the resulting equation in $(0, L)$, we obtain

$$
\begin{cases}
D_2\int\limits_0^L e^{\frac{q_h}{D_2}x}\varphi_{2x}\dot{\varphi}_{2x}\mathrm{d}x+\int\limits_0^L e^{\frac{q_h}{D_2}x}\delta_h(x)\varphi_2\dot{\varphi}_2\mathrm{d}x+\frac{q_h}{2D_2}\int\limits_0^L e^{\frac{q_h}{D_2}x}\varphi_2^2\mathrm{d}x \\
+\left(\nu_1-\frac{1}{2}\right)e^{\frac{q_h}{D_2}L}\varphi_2^2(L)+\nu_1 q_h e^{\frac{q_h}{D_2}L}\varphi_2(L)\dot{\varphi}_2(L)+\frac{1}{2}\varphi_2^2(0) \\
= \quad -\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}\int\limits_0^L e^{\frac{q_h}{D_2}x}k_1(x,P)\varphi_2\varphi_5\mathrm{d}x+\frac{1}{\mathcal{R}_0}\int\limits_0^L e^{\frac{q_h}{D_2}x}k_1(x,P)\varphi_2\dot{\varphi}_5\mathrm{d}x, \\
D_5\int\limits_0^L e^{\frac{q_v}{D_5}x}\varphi_{5x}\dot{\varphi}_{5x}\mathrm{d}x+\int\limits_0^L e^{\frac{q_v}{D_5}x}d_v(x)\varphi_5\dot{\varphi}_5\mathrm{d}x+\nu_2 q_v e^{\frac{q_v}{D_5}L}\varphi_5(L)\dot{\varphi}_5(L) \\
= \quad -\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}\int\limits_0^L e^{\frac{q_v}{D_5}x}k_2(x,A)\varphi_2\varphi_5\mathrm{d}x+\frac{1}{\mathcal{R}_0}\int\limits_0^L e^{\frac{q_v}{D_5}x}k_2(x,A)\dot{\varphi}_2\varphi_5\mathrm{d}x.
\end{cases}
\tag{4.2}
$$

Similarly, multiplying the first and second equations of (3.7) by $e^{q_h x/D_2}\dot{\varphi}_2$ and $e^{q_v x/D_5}\dot{\varphi}_5$, and then integrating the resulting equation over $(0, L)$ to give

$$
\begin{cases}
D_2\int\limits_0^L e^{\frac{q_h}{D_2}x}\varphi_{2x}\dot{\varphi}_{2x}\mathrm{d}x+\int\limits_0^L e^{\frac{q_h}{D_2}x}\delta_h(x)\varphi_2\dot{\varphi}_2\mathrm{d}x+\nu_1 q_h e^{\frac{q_h}{D_2}L}\varphi_2(L)\dot{\varphi}_2(L) \\
= \quad \frac{1}{\mathcal{R}_0}\int\limits_0^L e^{\frac{q_h}{D_2}x}k_1(x,P)\dot{\varphi}_2\varphi_5\mathrm{d}x, \\
D_5\int\limits_0^L e^{\frac{q_v}{D_5}x}\varphi_{5x}\dot{\varphi}_{5x}\mathrm{d}x+\int\limits_0^L e^{\frac{q_v}{D_5}x}d_v(x)\varphi_5\dot{\varphi}_5\mathrm{d}x+\nu_2 q_v e^{\frac{q_v}{D_5}L}\varphi_5(L)\dot{\varphi}_5(L) \\
= \quad \frac{1}{\mathcal{R}_0}\int\limits_0^L e^{\frac{q_v}{D_5}x}k_2(x,A)\dot{\varphi}_2\varphi_5\mathrm{d}x.
\end{cases}
\tag{4.3}
$$

Subtracting (4.2) from (4.3) to get

$$
\begin{cases}
\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}\int\limits_0^L e^{\frac{q_h}{D_2}x}k_1(x,P)\varphi_2\varphi_5\mathrm{d}x = & -\left(\nu_1-\frac{1}{2}\right)e^{\frac{q_h}{D_2}L}\varphi_2^2(L)-\frac{1}{2}\varphi_2^2(0)-\frac{q_h}{2D_2}\int\limits_0^L e^{\frac{q_h}{D_2}x}\varphi_2^2\mathrm{d}x \\
& +\frac{1}{\mathcal{R}_0}\int\limits_0^L e^{\frac{q_h}{D_2}x}k_1(x,P)(\varphi_2\dot{\varphi}_5-\dot{\varphi}_2\varphi_5)\mathrm{d}x, \\
\frac{\dot{\mathcal{R}}_0}{\mathcal{R}_0^2}\int\limits_0^L e^{\frac{q_v}{D_5}x}k_2(x,A)\varphi_5\varphi_2\mathrm{d}x = & \frac{1}{\mathcal{R}_0}\int\limits_0^L e^{\frac{q_v}{D_5}x}k_2(x,A)(\dot{\varphi}_2\varphi_5-\varphi_2\dot{\varphi}_5)\mathrm{d}x.
\end{cases}
$$

By adding the above two equalities, one has

$$\dot{\mathcal{R}}_0 = -\mathcal{R}_0^2 \frac{\left(\nu_1 - \frac{1}{2}\right) e^{\frac{q_h}{D_2}L} \varphi_2^2(L) + \frac{1}{2}\varphi_2^2(0) + \frac{q_h}{2D_2} \int_0^L e^{\frac{q_h}{D_2}x} \varphi_2^2 \mathrm{d}x}{\int_0^L e^{\frac{q_h}{D_2}x} [k_1(x,P) + k_2(x,A)] \varphi_2 \varphi_5 \mathrm{d}x} < 0$$

which is owing to the assumption (F2) and $k_1(\cdot, P) \equiv k_2(\cdot, A)$ in $(0, L)$, $\nu_1 \geq 1/2$. Therefore, $\mathcal{R}_0$ is a strictly monotone decreasing function of q_h. Through the similar arguments, $\mathcal{R}_0$ is a strictly monotone decreasing function of q_v. In a similar fashion, we can show the monotonicity of $\mathcal{R}_0$ for system (1.3). This ends the proof. $\square$

Remark 4.1. In biology, the condition $\nu_i \geq 1/2$ indicates that only when the infected loss of the host or vector at the downstream end has to be at least half of the advective effect, so as to guarantee the monotonicity of $\mathcal{R}_0$ w.r.t q_h or q_v.

In what follows, we study the asymptotic profiles of $\mathcal{R}_0$ of systems (1.2) and (1.3).

Proposition 4.2. *Suppose the conditions of Lemma 3.1 hold. Then, for $i = 1, 2$,*

(i) *For each $D_2 > 0$, $D_5 > 0$ and $\nu_i \in [1/2, \infty]$, $\mathcal{R}_0 \to 0$ as $q_h \to \infty$ or $q_v \to \infty$;*
(ii) *For each $q_h > 0$, $q_v > 0$ and $\nu_i \in [1/2, \infty]$, $\mathcal{R}_0 \to 0$ as $D_2 \to 0$ or $D_5 \to 0$;*
(iii) *For each $q_h > 0$, $q_v > 0$ and $\nu_i \in (0, \infty)$, $\mathcal{R}_0 \to \sqrt{\mathcal{R}_1^a \mathcal{R}_2^a}$ as $D_2 \to \infty$ and $D_5 \to \infty$, where*

$$\mathcal{R}_1^a := \frac{\int_0^L k_1(x,P)\mathrm{d}x}{\nu_1 q_h + \int_0^L \delta_h(x)\mathrm{d}x}, \quad \mathcal{R}_2^a := \frac{\int_0^L k_2(x,A)\mathrm{d}x}{\nu_2 q_v + \int_0^L d_v(x)\mathrm{d}x};$$

(iv) *For each $D_2 > 0$, $D_5 > 0$ and $\nu_i \in (0, \infty]$, $\mathcal{R}_0 \to \widetilde{\mathcal{R}}_0$ as $q_h \to 0$ and $q_v \to 0$, where $\widetilde{\mathcal{R}}_0$ denotes the basic reproduction ratio of (1.2) for $q_h = q_v = 0$ which is given by*

$$\widetilde{\mathcal{R}}_0 = \sup_{\substack{\varphi_2, \varphi_5 \in H^1((0,L)) \\ \varphi_2 \neq 0,\, \varphi_5 \neq 0}} \left\{ \sqrt{\widetilde{\Theta}_1^0(\varphi_2,\varphi_5)\widetilde{\Theta}_2^0(\varphi_2,\varphi_5)} \right\}, \tag{4.4}$$

where

$$\widetilde{\Theta}_1^0(\varphi_2,\varphi_5) := \frac{\int_0^L k_1(x,P_0)\varphi_2\varphi_5 \mathrm{d}x}{D_2 \int_0^L \varphi_{2x}^2 \mathrm{d}x + \int_0^L \delta_h(x)\varphi_2^2 \mathrm{d}x}, \quad \widetilde{\Theta}_2^0(\varphi_2,\varphi_5) := \frac{\int_0^L k_2(x,A_0)\varphi_2\varphi_5 \mathrm{d}x}{D_5 \int_0^L \varphi_{5x}^2 \mathrm{d}x + \int_0^L d_v(x)\varphi_5^2 \mathrm{d}x},$$

and P_0 (resp. A_0) is the unique positive steady state of (2.1) for $D = D_1$, $\mu(x) = d_h(x)$ (resp. $D = D_4$, $\mu(x) = d_v(x)$) and $q = 0$.

Proof. For $\nu_i \in (0,\infty)$, $i=1,2$. By (3.5), one has

$$
\begin{aligned}
\mathcal{R}_0^2 = & \sup_{\substack{\varphi_2,\varphi_5\in H^1((0,L))\\ \varphi_2\neq 0,\ \varphi_5\neq 0}} \{\Theta_1(\varphi_2,\varphi_5)\Theta_2(\varphi_2,\varphi_5)\} \\
\geq & \frac{\int_0^L k_1(x,P)e^{\frac{q_h}{D_2}x}\mathrm{d}x \int_0^L k_2(x,A)e^{\frac{q_v}{D_5}x}\mathrm{d}x}{\left(\nu_1 q_h e^{\frac{q_h}{D_2}L} + \int_0^L \delta_h(x)e^{\frac{q_h}{D_2}x}\mathrm{d}x\right)\left(\nu_2 q_v e^{\frac{q_v}{D_5}L} + \int_0^L d_v(x)e^{\frac{q_v}{D_5}x}\mathrm{d}x\right)} \\
\geq & \frac{\int_0^L k_1(x,P)\mathrm{d}x \int_0^L k_2(x,A)\mathrm{d}x}{e^{(\frac{q_h}{D_2}+\frac{q_v}{D_5})L}\left(\nu_1 q_h + \int_0^L \delta_h(x)\mathrm{d}x\right)\left(\nu_2 q_v + \int_0^L d_v(x)\mathrm{d}x\right)}
\end{aligned}
\tag{4.5}
$$

which is owing to the fact that $e^{q_h x/D_2} \geq 1$ and $e^{q_v x/D_5} \geq 1$, $x \in [0,L]$. Then $\mathcal{R}_0$ is bounded below for sufficiently large q_h or q_v or sufficiently small D_2 or D_5. Inspired by the arguments in [15, Lemma 4.9], letting $(\varphi_2(x),\varphi_5(x)) = e^{-\alpha x/2}(\phi_2(x),\phi_5(x))$ in (3.5), where $\alpha = q_h/D_2 = q_v/D_5$, and with the aid of the hypothesis $\nu_i \geq 1/2$, $i=1,2$, one gets

$$
\begin{aligned}
& \nu_1 q_h e^{\frac{q_h}{D_2}L}\varphi_2^2(L) + D_2\int_0^L e^{\frac{q_h}{D_2}x}\varphi_{2x}^2\mathrm{d}x + \int_0^L \delta_h(x)e^{\frac{q_h}{D_2}x}\varphi_2^2\mathrm{d}x \\
= & \nu_1 q_h \phi_2^2(L) + D_2\int_0^L \left(-\frac{\alpha}{2}\phi_2 + \phi_{2x}\right)^2\mathrm{d}x + \int_0^L \delta_h(x)\phi_2^2\mathrm{d}x \\
= & \nu_1 q_h \phi_2^2(L) + \frac{q_h^2}{4D_2}\int_0^L \phi_2^2\mathrm{d}x - q_h\int_0^L \phi_2\phi_{2x}\mathrm{d}x + D_2\int_0^L \phi_{2x}^2\mathrm{d}x + \int_0^L \delta_h(x)\phi_2^2\mathrm{d}x \\
= & \left(\nu_1 - \frac{1}{2}\right)q_h\phi_2^2(L) + \frac{q_h}{2}\phi_2^2(0) + \frac{q_h^2}{4D_2}\int_0^L \phi_2^2\mathrm{d}x + D_2\int_0^L \phi_{2x}^2\mathrm{d}x + \int_0^L \delta_h(x)\phi_2^2\mathrm{d}x, \\
\geq & \left(\frac{q_h^2}{4D_2} + \delta_h^-\right)\int_0^L \phi_2^2\mathrm{d}x,
\end{aligned}
$$

and

$$
\nu_2 q_v e^{\frac{q_v}{D_5}L}\varphi_5^2(L) + D_5\int_0^L e^{\frac{q_v}{D_5}x}\varphi_{5x}^2\mathrm{d}x + \int_0^L d_v(x)e^{\frac{q_v}{D_5}x}\varphi_5^2\mathrm{d}x
$$

$$
\begin{aligned}
&=\left(\nu_2-\frac{1}{2}\right)q_v\phi_5^2(L)+\frac{q_v}{2}\phi_5^2(0)+\frac{q_v^2}{4D_5}\int_0^L\phi_5^2\mathrm{d}x+D_5\int_0^L\phi_{5x}^2\mathrm{d}x+\int_0^L d_v(x)\phi_5^2\mathrm{d}x\\
&\ge\left(\frac{q_v^2}{4D_5}+d_v^-\right)\int_0^L\phi_5^2\mathrm{d}x.
\end{aligned}
$$

In addition, by means of the Cauchy-Schwarz inequality, we have

$$
\int_0^L k_1(x,P)e^{\frac{q_h}{D_2}x}\varphi_2\varphi_5\mathrm{d}x=\int_0^L k_1(x,P)\phi_2\phi_5\mathrm{d}x\le k_1^+\left(\int_0^L\phi_2^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^L\phi_5^2\mathrm{d}x\right)^{\frac{1}{2}},
$$

and

$$
\int_0^L k_2(x,A)e^{\frac{q_v}{D_5}x}\varphi_2\varphi_5\mathrm{d}x\le k_2^+\left(\int_0^L\phi_2^2\mathrm{d}x\right)^{\frac{1}{2}}\left(\int_0^L\phi_5^2\mathrm{d}x\right)^{\frac{1}{2}}.
$$

As a consequence,

$$
\begin{aligned}
\frac{1}{\mathcal{R}_0^2}&=\inf_{\substack{\varphi_2,\varphi_5\in H^1((0,L))\\ \varphi_2\neq0,\varphi_5\neq0}}\left\{\frac{1}{\Theta_1(\varphi_2,\varphi_5)\Theta_2(\varphi_2,\varphi_5)}\right\}\\
&\ge\inf_{\substack{\phi_2,\phi_5\in H^1((0,L))\\ \phi_2\neq0,\phi_5\neq0}}\left\{\frac{\left(\frac{q_h^2}{4D_2}+\delta_h^-\right)\left(\frac{q_v^2}{4D_5}+d_v^-\right)\int_0^L\phi_2^2\mathrm{d}x\int_0^L\phi_5^2\mathrm{d}x}{k_1^+k_2^+\int_0^L\phi_2^2\mathrm{d}x\int_0^L\phi_5^2\mathrm{d}x}\right\}\\
&=\frac{\left(\frac{q_h^2}{4D_2}+\delta_h^-\right)\left(\frac{q_v^2}{4D_5}+d_v^-\right)}{k_1^+k_2^+}.
\end{aligned}
\tag{4.6}
$$

In light of (4.5)-(4.6), we see that (i) and (ii) hold when $\nu_i<\infty$, $i=1,2$. Similarly, one can show (i) and (ii) when $\nu_i=\infty$, $i=1,2$.

Since

$$
\begin{aligned}
\frac{1}{\mathcal{R}_0^2}&=\inf_{\substack{\varphi_2,\varphi_5\in H^1((0,L))\\ \varphi_2\neq0,\varphi_5\neq0}}\left\{\frac{1}{\Theta_1(\varphi_2,\varphi_5)\Theta_2(\varphi_2,\varphi_5)}\right\}\\
&\le\frac{\left[\nu_1q_he^{\frac{q_h}{D_2}L}+\int_0^L\delta_h(x)e^{\frac{q_h}{D_2}x}\mathrm{d}x\right]\left[\nu_2q_ve^{\frac{q_v}{D_5}L}+\int_0^L d_v(x)e^{\frac{q_v}{D_5}x}\mathrm{d}x\right]}{\int_0^L k_1(x,P)e^{\frac{q_h}{D_2}x}\mathrm{d}x\int_0^L k_2(x,A)e^{\frac{q_v}{D_5}x}\mathrm{d}x},
\end{aligned}
$$

it follows that $1/\mathcal{R}_0$ is uniformly bounded in terms of D_2 and D_5 large enough. Thus, passing to a sequence if necessary, there is a finite constant $\mathcal{R}_0^* > 0$ such that $\mathcal{R}_0 \to \mathcal{R}_0^*$ when $D_2 \to \infty$ and $D_5 \to \infty$. Let (ϖ_2, ϖ_5) be a positive eigenfunction corresponding to $1/\mathcal{R}_0$ of (3.3). Assume $\|\varpi_2\| + \|\varpi_5\| = 1$. Together with L^p estimate and the discussion of [26], $\|\varpi_2\|_{W^{2,p}((0,L))}$ and $\|\varpi_4\|_{W^{2,p}((0,L))}$ are uniformly bounded for any $p > 1$. And, thereby $\|\varpi_2\|_{C^{1,\varsigma}((0,L))}$ and $\|\varpi_5\|_{C^{1,\varsigma}((0,L))}$ are uniformly bounded for any $0 \le \varsigma \le 1$, by inspection of the Sobolev embedding theorem (see [8]). Then $\varpi_j(\cdot) \to \varpi_j^*$ in $C^1([0, L])$ for some $\varpi_j^* > 0$, $j = 2, 5$, as $D_2 \to \infty$ and $D_5 \to \infty$. The elliptic regularity estimate deduces that ϖ_j^* is a constant, $j = 2, 5$. Therefore, through integrating (3.3) over $(0, L)$ and letting $D_2 \to \infty$ and $D_5 \to \infty$, one obtains

$$\begin{cases} \nu_1 q_h \varpi_2^* + \varpi_2^* \int_0^L \delta_h(x)\mathrm{d}x = \dfrac{\varpi_5^*}{\mathcal{R}_0^*} \int_0^L k_1(x, P)\mathrm{d}x, \\ \nu_2 q_v \varpi_5^* + \varpi_5^* \int_0^L d_v(x)\mathrm{d}x = \dfrac{\varpi_2^*}{\mathcal{R}_0^*} \int_0^L k_2(x, A)\mathrm{d}x. \end{cases}$$

Then (iii) holds. For (iv), it is obvious and so we omit the details. This ends the proof. □

Remark 4.2. From Proposition 4.2 (i)-(ii), we can see that when the advection rate of infected hosts or vectors is dominant relative to their diffusion rates, the basic reproduction ratio of (1.2) (resp. (1.3)) tends to zero. Biologically, since the downstream environment is unfavorable for individual survival when $\nu_i \ge 1/2$, $i = 1, 2$, the infected hosts or vectors will be lost at the downstream end $x = L$ as q_h or q_h is large relative to D_2 or D_5. In this circumstance, the vector-borne disease will fade.

5. Classification on the dynamics of model (1.2)

Define

$$\mathcal{R}_{01}^{\mathrm{a}} := \sqrt{\mathcal{R}_{01}^{vh}\mathcal{R}_{01}^{hv}}, \quad \mathcal{R}_{02}^{\mathrm{a}} := \sqrt{\mathcal{R}_{02}^{vh}\mathcal{R}_{02}^{hv}},$$

where

$$\mathcal{R}_{01}^{vh} := \frac{\int_0^L k_1(x, P_0)\mathrm{d}x}{\int_0^L \delta_h(x)\mathrm{d}x}, \quad \mathcal{R}_{01}^{hv} := \frac{\int_0^L k_2(x, A_0)\mathrm{d}x}{\int_0^L d_v(x)\mathrm{d}x}, \quad \mathcal{R}_{02}^{vh} := \frac{\int_0^L k_1(x, P)\mathrm{d}x}{\int_0^L \delta_h(x)\mathrm{d}x}, \quad \mathcal{R}_{02}^{hv} := \frac{\int_0^L k_2(x, A)\mathrm{d}x}{\int_0^L d_v(x)\mathrm{d}x}.$$

We adopt the terminology analogous to that in [28]. The habitat is said to be a high-risk area if $\mathcal{R}_{01}^{\mathrm{a}} > 1$, and be a low-risk area if $\mathcal{R}_{01}^{\mathrm{a}} < 1$. In the following, we always fix $\nu_i \ge 1/2$, $i = 1, 2$.

5.1. *Classification on the dynamics in high-risk domain*

In this subsection, we explore the level set classification of $\mathcal{R}_0$ for (1.2) w.r.t the diffusion rates (D_2, D_5) and advection rates (q_h, q_v) in high-risk area.

Lemma 5.1. *Suppose the conditions of Lemma 3.1 hold, and $\mathcal{R}_0$ is defined by (3.5). If $\mathcal{R}_{01}^{\mathrm{a}} > 1$, then for each $D_2 > 0$ and $D_5 > 0$, there exist unique points $q_h^* = q_h^*(D_2, D_5)$ and $q_v^* = q_v^*(D_2, D_5)$ such that*

(i) *If $0 < q_h < q_h^*$ or $0 < q_v < q_v^*$, then $\mathcal{R}_0(D_2, D_5) > 1$;*
(ii) *If $q_h > q_h^*$ or $q_v > q_v^*$, then $\mathcal{R}_0(D_2, D_5) < 1$.*

Proof. By appealing to Proposition 4.2, for any $D_2 > 0$ and $D_5 > 0$, we have

$$\lim_{q_h \to 0,\, q_v \to 0} \mathcal{R}_0(D_2, D_5, q_h, q_v) = \widetilde{\mathcal{R}}_0(D_2, D_5),$$

and

$$\lim_{q_h \to \infty} \mathcal{R}_0(D_2, D_5, q_h, q_v) = 0, \quad \lim_{q_v \to \infty} \mathcal{R}_0(D_2, D_5, q_h, q_v) = 0.$$

Thanks to $\mathcal{R}_{01}^{\mathrm{a}} > 1$ and [28, Lemma 3.1], one has $\widetilde{\mathcal{R}}_0(D_2, D_5) > 1$. It then follows that there are at least two points $q_h^* = q_h^*(D_2, D_5)$ and $q_v^* = q_v^*(D_2, D_5)$ such that $\mathcal{R}_0(q_h^*, q_v^*) = 1$. Moreover, noting that $\mathcal{R}_0$ is strictly monotonically decreasing w.r.t q_h and q_v, respectively, in Proposition 4.1, hence (q_h^*, q_v^*) is unique. This completes the proof. □

Remark 5.1. By inspection of Lemma 5.1, it is straightforward to see that there are unique functions $q_h = \chi_1(D_2, D_5)$ and $q_v = \chi_2(D_2, D_5)$ such that $\mathcal{R}_0(D_2, D_5, \chi_1(D_2, D_5), \chi_2(D_2, D_5)) = 1$. In the sequel, we analyze the properties of χ_i, $i = 1, 2$.

Lemma 5.2. *Suppose the conditions of Lemma 3.1 hold. If $\min\{\mathcal{R}_{01}^{\mathrm{a}}, \mathcal{R}_{02}^{\mathrm{a}}\} > 1$, then for each $D_2 > 0$ and $D_5 > 0$, there exist unique functions $\chi_1(D_2, D_5)$ and $\chi_2(D_2, D_5)$: $(0, \infty)^2 \to (0, \infty)$ such that*

$$\lim_{D_2 \to 0^+,\, D_5 \to 0^+} \chi_i(D_2, D_5) = 0, \quad \lim_{D_2 \to \infty,\, D_5 \to \infty} \chi_i(D_2, D_5) = \theta_i^*, \ \ i = 1, 2,$$

where θ_1^* and θ_2^* satisfy $\theta_1^*/D_2 = \theta_2^*/D_5$, and θ_1^* is the unique positive solution of the equation

$$\left[\nu_1 \theta + \int_0^L \delta_h(x)\mathrm{d}x\right]\left[\nu_2 D_5 D_2^{-1}\theta + \int_0^L d_v(x)\mathrm{d}x\right] - \int_0^L k_1(x, P)\mathrm{d}x \int_0^L k_2(x, A)\mathrm{d}x = 0.$$

Proof. Passing to a subsequence if necessary, we assume that there exist two constants $p_i \in [0, \infty]$ such that $\chi_i(D_2, D_5) \to p_i$, $i = 1, 2$, as $D_2 \to 0^+$ and $D_5 \to 0^+$. If $p_1 = p_2 = \infty$, then there is a sufficiently small constant $a_s > 0$ and a sufficiently large constant $a_l > 0$ such that $\max\{\chi_1(D_2, D_5), \chi_2(D_2, D_5)\} > a_l$ when $\min\{D_2, D_5\} < a_s$. Hence, according to Proposition 4.2 (i), for fixed $\min\{D_2, D_5\} < a_s$, we have

$$\lim_{\chi_1(D_2, D_5) \to \infty,\, \chi_2(D_2, D_5) \to \infty} \mathcal{R}_0(D_2, D_5, \chi_1(D_2, D_5), \chi_2(D_2, D_5)) = 0,$$

which contradicts $\mathcal{R}_0(D_2, D_5, \chi_1(D_2, D_5), \chi_2(D_2, D_5)) = 1$ and thus $p_i \in [0, \infty)$, $i = 1, 2$. If $p_1, p_2 > 0$, then Proposition 4.2 (ii) yields that

$$\lim_{D_2\to 0^+, D_5\to 0^+, \chi_1(D_2,D_5)\to p_1, \chi_2(D_2,D_5)\to p_2} \mathcal{R}_0(D_2, D_5, \chi_1(D_2, D_5), \chi_2(D_2, D_5)) = 0,$$

which contradicts $\mathcal{R}_0(D_2, D_5, \chi_1(D_2, D_5), \chi_2(D_2, D_5)) = 1$ and so $p_1 = p_2 = 0$.

Similarly, supposing that there are two constants $\theta_i^* \in [0, \infty]$ such that $\chi_i(D_2, D_5) \to \theta_i^*$, $i = 1, 2$, as $D_2 \to \infty$ and $D_5 \to \infty$. If $\theta_1^* = \theta_2^* = \infty$, then there are sufficiently large positive constants a_{l1} and a_{l2} such that $\max\{\chi_1(D_2, D_5), \chi_2(D_2, D_5)\} > a_{l2}$ when $\max\{D_2, D_5\} > a_{l1}$. Hence, by Proposition 4.2 (i), for fixed $\max\{D_2, D_5\} > a_{l1}$, one gets

$$\lim_{\chi_1(D_2,D_5)\to\infty,\ \chi_2(D_2,D_5)\to\infty} \mathcal{R}_0(D_2, D_5, \chi_1(D_2, D_5), \chi_2(D_2, D_5)) = 0,$$

which also contradicts $\mathcal{R}_0(D_2, D_5, \chi_1(D_2, D_5), \chi_2(D_2, D_5)) = 1$ and then $\theta_i^* \in [0, \infty)$, $i = 1, 2$.

To illustrate $\theta_i^* > 0$, $i = 1, 2$. Let $(\varphi_2^*, \varphi_5^*)$ be the positive eigenfunction corresponding to $\mathcal{R}_0(D_2, D_5, \chi_1(D_2, D_5), \chi_2(D_2, D_5)) = 1$ (i.e., $\varrho_0 = 1$) of problem (3.7) satisfying $\|\varphi_2^*\| + \|\varphi_5^*\| = 1$. Multiplying the two equations of (3.7) by $e^{\chi_1(D_2,D_5)x/D_2}$ and $e^{\chi_2(D_2,D_5)x/D_5}$ to give

$$\begin{cases} -D_2\left[e^{\frac{\chi_1(D_2,D_5)}{D_2}x}\varphi_{2x}^*\right]_x = e^{\frac{\chi_1(D_2,D_5)}{D_2}x}[-\delta_h(x)\varphi_2^* + k_1(x,P)\varphi_5^*], & 0<x<L, \\ -D_5\left[e^{\frac{\chi_2(D_2,D_5)}{D_5}x}\varphi_{5x}^*\right]_x = e^{\frac{\chi_2(D_2,D_5)}{D_5}x}[-d_v(x)\varphi_5^* + k_2(x,A)\varphi_2^*], & 0<x<L, \\ \varphi_{2x}^*(0) = \varphi_{5x}^*(0) = 0, \\ D_2\varphi_{2x}^*(L) + \nu_1 q_h \varphi_2^*(L) = D_5\varphi_{5x}^*(L) + \nu_2 q_v \varphi_5^*(L) = 0. \end{cases}$$

By integrating the above equality on $(0, L)$, we obtain

$$\begin{cases} \nu_1\chi_1(D_2,D_5)e^{\frac{\chi_1(D_2,D_5)}{D_2}L}\varphi_2^*(L) + \int_0^L e^{\frac{\chi_1(D_2,D_5)}{D_2}x}\delta_h(x)\varphi_2^*\mathrm{d}x - \int_0^L e^{\frac{\chi_1(D_2,D_5)}{D_2}x}k_1(x,P)\varphi_5^*\mathrm{d}x = 0, \\ \nu_2\chi_2(D_2,D_5)e^{\frac{\chi_2(D_2,D_5)}{D_5}L}\varphi_5^*(L) + \int_0^L e^{\frac{\chi_2(D_2,D_5)}{D_5}x}d_v(x)\varphi_5^*\mathrm{d}x - \int_0^L e^{\frac{\chi_2(D_2,D_5)}{D_5}x}k_2(x,A)\varphi_2^*\mathrm{d}x = 0. \end{cases} \tag{5.1}$$

With the aid of the standard elliptic regularity estimate, there exist positive constants $\bar{\varphi}_2^*$ and $\bar{\varphi}_5^*$ satisfying $\|\bar{\varphi}_2^*\| + \|\bar{\varphi}_5^*\| = 1$ such that $(\varphi_2^*, \varphi_5^*) \to (\bar{\varphi}_2^*, \bar{\varphi}_5^*)$ as $D_2 \to \infty$ and $D_5 \to \infty$. Taking a limit by letting $D_2 \to \infty$ and $D_5 \to \infty$ in (5.1), it follows that

$$\begin{pmatrix} \nu_1\theta_1^* + \int_0^L \delta_h(x)\mathrm{d}x & -\int_0^L k_1(x,P)\mathrm{d}x \\ -\int_0^L k_2(x,A)\mathrm{d}x & \nu_2\theta_2^* + \int_0^L \delta_h(x)\mathrm{d}x \end{pmatrix}\begin{pmatrix} \bar{\varphi}_2^* \\ \bar{\varphi}_5^* \end{pmatrix} := \mathcal{K}\begin{pmatrix} \bar{\varphi}_2^* \\ \bar{\varphi}_5^* \end{pmatrix} = 0.$$

Since $\bar{\varphi}_2^*$ and $\bar{\varphi}_5^*$ are constants, the determinant of matrix $\mathcal{K}$ vanishes, that is,

$$\left[\nu_1\theta_1^* + \int_0^L \delta_h(x)\mathrm{d}x\right]\left[\nu_2\theta_2^* + \int_0^L \delta_h(x)\mathrm{d}x\right] - \int_0^L k_1(x,P)\mathrm{d}x \int_0^L k_2(x,A)\mathrm{d}x = 0.$$

From the assumption (F2), we get $\theta_1^*/D_2 = \theta_2^*/D_5$ and thus

$$\left[\nu_1\theta_1^* + \int_0^L \delta_h(x)\mathrm{d}x\right]\left[\nu_2 D_5 D_2^{-1}\theta_1^* + \int_0^L d_v(x)\mathrm{d}x\right] - \int_0^L k_1(x,P)\mathrm{d}x \int_0^L k_2(x,A)\mathrm{d}x = 0. \quad (5.2)$$

By $\mathcal{R}_{02}^{\mathrm{a}} > 1$, it is easy to verify that (5.2) admits a unique root $\theta_1^* > 0$. This ends the proof. □

Then we have the following main conclusions.

Theorem 5.1. *Suppose the conditions of Lemma 3.1 hold. If* $\min\{\mathcal{R}_{01}^{\mathrm{a}}, \mathcal{R}_{02}^{\mathrm{a}}\} > 1$, *then for each* $D_2 > 0$ *and* $D_5 > 0$, *then there are unique surfaces*

$$\Omega_1 = \{(q_h, \chi_1(D_2, D_5)) : \mathcal{R}_0(D_2, D_5, \chi_1(D_2, D_5)) = 1, (D_2, D_5) \in (0,\infty)^2\},$$

and

$$\Omega_2 = \{(q_v, \chi_2(D_2, D_5)) : \mathcal{R}_0(D_2, D_5, \chi_2(D_2, D_5)) = 1, (D_2, D_5) \in (0,\infty)^2\},$$

in spaces $q_h - (D_2, D_5)$ *and* $q_v - (D_2, D_5)$, *respectively, such that system* (1.2) *is uniformly persistent and admits at least one EE for any* $0 < q_h < \chi_1(D_2, D_5)$ *or* $0 < q_v < \chi_2(D_2, D_5)$, *and* E_0 *is GAS for any* $q_h > \chi_1(D_2, D_5)$ *or* $q_v > \chi_2(D_2, D_5)$. *Furthermore,* $\chi_1(D_2, D_5)$ *and* $\chi_2(D_2, D_5)$: $(0,\infty)^2 \to (0,\infty)$ *fulfill*

$$\lim_{D_2\to 0^+,\, D_5\to 0^+} \chi_i(D_2, D_5) = 0, \quad \lim_{D_2\to\infty,\, D_5\to\infty} \chi_i(D_2, D_5) = \theta_i^*, \ \ i = 1, 2,$$

where θ_1^* *and* θ_2^* *satisfy* $\theta_1^*/D_2 = \theta_2^*/D_5$, *and* θ_1^* *is the unique positive solution of the equation*

$$\left[\nu_1\theta + \int_0^L \delta_h(x)\mathrm{d}x\right]\left[\nu_2 D_5 D_2^{-1}\theta + \int_0^L d_v(x)\mathrm{d}x\right] - \int_0^L k_1(x,P)\mathrm{d}x \int_0^L k_2(x,A)\mathrm{d}x = 0.$$

Remark 5.2. Define

$$\Pi_{q_j}^S = \{(D_2, D_5, q_j) : \mathcal{R}_0(D_2, D_5, q_j) < 1,\ \mathcal{R}_{01}^{\mathrm{a}} > 1,\ \mathcal{R}_{02}^{\mathrm{a}} > 1\},$$

and

$$\Pi_{q_j}^U = \{(D_2, D_5, q_j) : \mathcal{R}_0(D_2, D_5, q_j) > 1,\ \mathcal{R}_{01}^{\mathrm{a}} > 1,\ \mathcal{R}_{02}^{\mathrm{a}} > 1\},\ \ j \in \{h, v\}.$$

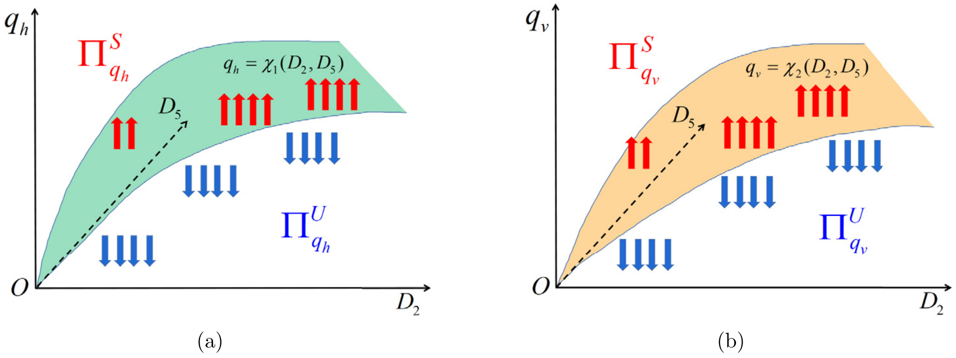


Fig. 1. Dynamic classification of model (1.2) in Theorem 5.1. The direction of red and blue arrows denotes the areas $\Pi^S_{q_j}$ and $\Pi^U_{q_j}$, respectively. Namely, $\Pi^S_{q_j} = \{(D_2, D_5, q_j) : q_j > \chi_i(D_2, D_5), D_2 > 0, D_5 > 0\}$ and $\Pi^U_{q_j} = \{(D_2, D_5, q_j) : 0 < q_j < \chi_i(D_2, D_5), D_2 > 0, D_5 > 0\}$, $i = 1, 2$, $j \in \{h, v\}$. (a) In space $q_h - (D_2, D_5)$, E_0 is GAS when $(D_2, D_5, q_h) \in \Pi^S_{q_h}$ which means that the disease extinction happens, while system (1.2) is uniformly persistent when $(D_2, D_5, q_h) \in \Pi^U_{q_h}$ which implies that the disease will surge; (b) In space $q_v - (D_2, D_5)$, E_0 is GAS when $(D_2, D_5, q_v) \in \Pi^S_{q_v}$, while system (1.2) is uniformly persistent when $(D_2, D_5, q_v) \in \Pi^U_{q_v}$. (For interpretation of the colors in the figure(s), the reader is referred to the web version of this article.)

The illustrations of Theorem 5.1 are given by Fig. 1.

The conclusions of Theorem 5.1 show that when $\min\{\mathcal{R}^{\mathrm{a}}_{01}, \mathcal{R}^{\mathrm{a}}_{02}\} > 1$, there are two thresholds w.r.t advection rates, such that when the advection rate of infected hosts or vectors exceeds the threshold, the disease will go extinct, but when it falls below the threshold, the disease will break out. Biologically, in open advective environments, when the advection rate of hosts or vectors is large enough, the hosts or vectors will be washed out from the habitat, resulting in the extinction of the disease.

5.2. *Classification on the dynamics in low-risk domain*

In the following, we discuss the level set classification of $\mathcal{R}_0$ in low-risk areas.

Lemma 5.3. *Suppose the conditions of Lemma 3.1 hold, and $\mathcal{R}_0$ is defined by (3.5). If $\mathcal{R}^{\mathrm{a}}_{01} < 1$, and $k_1(x, P_0)k_2(x, A_0) - \delta_h(x)d_v(x)$ changes sign in $(0, L)$, then there exist two positive constants D^*_2 and D^*_5, which is the unique root of the equation $\widetilde{\mathcal{R}}_0(D_2, D_5) = 1$, such the following assertions hold:*

(i) *For $D_2 \in (0, D^*_2)$ and $D_5 \in (0, D^*_5)$, there exist unique points $q^{**}_h = q^{**}_h(D_2, D_5)$ and $q^{**}_v = q^{**}_v(D_2, D_5)$, such that $\mathcal{R}_0(D_2, D_5) > 1$ for any $0 < q_h < q^{**}_h$ or $0 < q_v < q^{**}_v$, and $\mathcal{R}_0(D_2, D_5) < 1$ for any $q_h > q^{**}_h$ or $q_v > q^{**}_v$;*
(ii) *For $D_2 \in [D^*_2, \infty)$ and $D_5 \in [D^*_5, \infty)$, $\mathcal{R}_0(D_2, D_5) < 1$ for any $q_h > 0$ and $q_v > 0$.*

Proof. It follows from Proposition 4.2 that

$$\lim_{q_h \to \infty} \mathcal{R}_0(D_2, D_5, q_h, q_v) = 0, \quad \lim_{q_v \to \infty} \mathcal{R}_0(D_2, D_5, q_h, q_v) = 0.$$

Since $\mathcal{R}_{01}^{a} < 1$ and $k_1(x, P_0)k_2(x, A_0) - \delta_h(x)d_v(x)$ changes sign in $(0, L)$, the Lemma 3.1 in [28] implies that the equation $\widetilde{\mathcal{R}}_0(D_2, D_5) = 1$ possesses a unique positive root pair (D_2^*, D_5^*) such that $\widetilde{\mathcal{R}}_0(D_2, D_5) > 1$ for $D_2 \in (0, D_2^*)$ and $D_5 \in (0, D_5^*)$, and $\widetilde{\mathcal{R}}_0(D_2, D_5) \le 1$ for $D_2 \in [D_2^*, \infty)$ and $D_5 \in [D_5^*, \infty)$. Thus, in light of the monotonicity of $\mathcal{R}_0$ w.r.t q_h and q_v, and the fact

$$\lim_{q_h \to 0,\, q_v \to 0} \mathcal{R}_0(D_2, D_5, q_h, q_v) = \widetilde{\mathcal{R}}_0(D_2, D_5),$$

we obtain that when $D_2 \in (0, D_2^*)$ and $D_5 \in (0, D_5^*)$, there exist the unique points q_h^{**} and q_v^{**} such that $\mathcal{R}_0(D_2, D_5, q_h, q_v) > 1$ for $q_h \in (0, q_h^{**})$ or $q_v \in (0, q_v^{**})$ and $\mathcal{R}_0(D_2, D_5, q_h, q_v) < 1$ for $q_h \in (q_h^{**}, \infty)$ or $q_v \in (q_v^{**}, \infty)$; when $D_2 \in [D_2^*, \infty)$ and $D_5 \in [D_5^*, \infty)$, $\mathcal{R}_0(D_2, D_5, q_h, q_v) < 1$ for any $q_h > 0$ and $q_v > 0$. □

Remark 5.3. In view of Lemma 5.3, there exist unique functions $q_h = \chi_3(D_2, D_5)$ and $q_v = \chi_4(D_2, D_5)$ such that $\mathcal{R}_0(D_2, D_5, \chi_3(D_2, D_5), \chi_4(D_2, D_5)) = 1$ when $D_2 \in (0, D_2^*)$ and $D_5 \in (0, D_5^*)$. Furthermore, χ_i has the following properties, $i = 3, 4$.

Lemma 5.4. *Suppose the conditions of Lemma 3.1 hold. If $\mathcal{R}_{01}^{a} < 1$, and $k_1(x, P_0)k_2(x, A_0) - \delta_h(x)d_v(x)$ changes sign on $(0, L)$, then there exist unique functions $\chi_3(D_2, D_5)$ and $\chi_4(D_2, D_5)$: $(0, D_2^*) \times (0, D_5^*) \to (0, \infty)$ such that*

$$\lim_{D_2 \to 0^+,\, D_5 \to 0^+} \chi_i(D_2, D_5) = 0, \quad \lim_{D_2 \to D_2^{*-},\, D_5 \to D_5^{*-}} \chi_i(D_2, D_5) = 0, \;\; i = 3, 4.$$

Proof. Similar to arguments of Lemma 5.2, one has $\chi_i(D_2, D_5) \to 0$ as $D_2 \to 0^+$ and $D_5 \to 0^+$, $i = 3, 4$. To show $\chi_i(D_2, D_5) \to 0$ as $D_2 \to D_2^{*-}$ and $D_5 \to D_5^{*-}$, $i = 3, 4$. Assume that there are some positive constants $c_0 \le \infty$ and $c_1 \le \infty$ such that $\chi_3(D_2, D_5) \to c_0$ and $\chi_4(D_2, D_5) \to c_1$ as $D_2 \to D_2^{*-}$ and $D_5 \to D_5^{*-}$. To substantiate that c_0 and c_1 are finite. Suppose not. Then $\mathcal{R}_0(D_2, D_5, \chi_3(D_2, D_5), \chi_4(D_2, D_5)) \to 0$ as $D_2 \to D_2^{*-}$ and $D_5 \to D_5^{*-}$ due to Proposition 4.2 which contradicts $\mathcal{R}_0(D_2, D_5, \chi_3(D_2, D_5), \chi_4(D_2, D_5)) = 1$. By the monotonicity of $\mathcal{R}_0$ w.r.t q_h and q_v, $1 = \mathcal{R}_0(D_2^*, D_5^*, c_0, c_1) < \mathcal{R}_0(D_2^*, D_5^*, 0, 0) = 1$ which is a contradiction. Hence, $c_0 = 0$ and $c_1 = 0$. This completes the proof. □

Consequently, we obtain the main results.

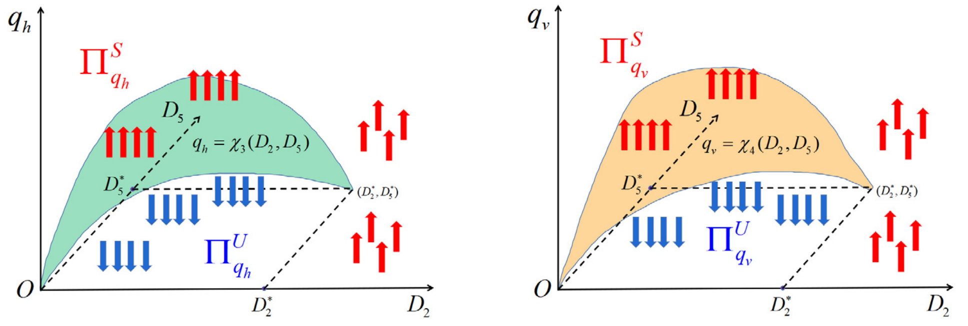
Theorem 5.2. *Suppose the conditions of Lemma 3.1 hold. If $\mathcal{R}_{01}^{a} < 1$ and $k_1(x, P_0)k_2(x, A_0) - \delta_h(x)d_v(x)$ changes sign in $(0, L)$, then there exist two positive constants D_2^* and D_5^*, which is the unique root of the equation $\widetilde{\mathcal{R}}_0(D_2, D_5) = 1$, such the following properties hold:*

(i) *For $D_2 \in (0, D_2^*)$ and $D_5 \in (0, D_5^*)$, there exist unique surfaces*

$$\Omega_3 = \{(q_h, \chi_3(D_2, D_5)) : \mathcal{R}_0(D_2, D_5, \chi_3(D_2, D_5)) = 1, (D_2, D_5) \in (0, D_2^*) \times (0, D_5^*)\},$$

and

$$\Omega_4 = \{(q_v, \chi_4(D_2, D_5)) : \mathcal{R}_0(D_2, D_5, \chi_4(D_2, D_5)) = 1, (D_2, D_5) \in (0, D_2^*) \times (0, D_5^*)\},$$



(a) (b)

Fig. 2. Dynamic classification of model (1.2) in Theorem 5.2. The direction of red and blue arrows represents the regions $\Pi^S_{q_j}$ and $\Pi^U_{q_j}$, respectively. In other words, $\Pi^S_{q_j} = \Pi^{S-1}_{q_j} \cup \Pi^{S-2}_{q_j}$ where $\Sigma^{S-1}_{q_j} = \{(D_2, D_5, q_j) : q_j > \chi_j(D_2, D_5), (D_2, D_5) \in (0, D_2^*) \times (0, D_5^*)\}$ and $\Pi^{S-2}_{q_j} = \{(D_2, D_5, q_j) : q_j > 0, (D_2, D_5) \in [D_2^*, \infty) \times [D_5^*, \infty)\}$, and $\Pi^U_{q_j} = \{(D_2, D_5, q_j) : 0 < q_j < \chi_i(D_2, D_5), (D_2, D_5) \in (0, D_2^*) \times (0, D_5^*)\}$, $i = 3, 4$, $j \in \{h, v\}$. (a) In space $q_h - (D_2, D_5)$, E_0 is GAS when $(D_2, D_5, q_h) \in \Pi^S_{q_h}$ which implies that the disease will disappear, while system (1.2) is uniformly persistent when $(D_2, D_5, q_h) \in \Pi^U_{q_h}$ which means that the disease will surge; (b) In space $q_v - (D_2, D_5)$, E_0 is GAS when $(D_2, D_5, q_v) \in \Pi^S_{q_v}$, while system (1.2) is uniformly persistent when $(D_2, D_5, q_v) \in \Pi^U_{q_v}$.

in spaces $q_h - (D_2, D_5)$ and $q_v - (D_2, D_5)$, respectively, such that system (1.2) *is uniformly persistent and admits at least one EE for any $0 < q_h < \chi_3(D_2, D_5)$ or $0 < q_v < \chi_4(D_2, D_5)$, and E_0 is GAS for any $q_h > \chi_3(D_2, D_5)$ or $q_v > \chi_4(D_2, D_5)$. Furthermore, $\chi_i(D_I, d_I) : (0, D_2^*) \times (0, D_5^*) \to (0, \infty)$ fulfills*

$$\lim_{D_2 \to 0^+,\, D_5 \to 0^+} \chi_i(D_2, D_5) = 0, \qquad \lim_{D_2 \to D_2^{*-},\, D_5 \to D_5^{*-}} \chi_i(D_2, D_5) = 0, \quad i = 3, 4.$$

(ii) *For $D_2 \in [D_2^*, \infty)$ and $D_5 \in [D_5^*, \infty)$, E_0 is GAS for any $q_h > 0$ and $q_v > 0$.*

Remark 5.4. Define

$$\Pi^S_{q_j} = \{(D_2, D_5, q_j) : \mathcal{R}_0(D_2, D_5, q_j) < 1,\ \mathcal{R}^{\mathrm{a}}_{01} < 1\},$$

and

$$\Pi^U_{q_j} = \{(D_2, D_5, q_j) : \mathcal{R}_0(D_2, D_5, q_j) > 1,\ \mathcal{R}^{\mathrm{a}}_{01} < 1\}, \quad j \in \{h, v\}.$$

The descriptions of Theorem 5.2 are illustrated as in Fig. 2.

Let $\overline{\chi}_i := \max\{\chi_i(D_2, D_5) : (D_2, D_5) \in [0, D_2^*] \times [0, D_5^*]\}$, $i = 3, 4$. Theorem 5.2 (i) implies that: As long as the advection rate is large enough to make $q_h > \overline{\chi}_3$ (or $q_v > \overline{\chi}_4$), no matter what the dispersal rate is, the disease will fade. Note that the stability of E_0 will change at least twice with the increase of D_2 and D_5 if $0 < q_h < \overline{\chi}_3$ (or $0 < q_v < \overline{\chi}_4$) is fixed. That is to say, E_0 is GAS when D_2 and D_5 are small or large enough, but system (1.2) is uniformly persistent when D_2 and D_5 are between some intermediate values. In biology, for sufficiently small diffusion, advection effects convey hosts or vectors to a disadvantageous place since the hosts or vectors will be washed out from the habitat at the downstream end. For sufficiently large diffusion, recalling that the habit is a low-risk site, the disease will also be eliminated. It can be observed from Theorems 5.1 and 5.2 that the use of open advective environment in vector-borne disease modeling will produce novel and interesting disease dynamics.

6. Conclusion and discussion

As a continuation of reference [28], this paper further studied a spatial vector-borne disease model with general incidences and general boundary conditions. Owing to the boundary conditions, we first applied the eigenvalue theory of elliptic system to investigate the existence of DFE (Proposition 2.1), which allows us to discuss the global existence and ultimate boundedness with the help of the classical induction method (Theorem 2.1). Moreover, the threshold dynamics were also investigated, i.e., DFE is GAS when $\mathcal{R}_0 < 1$ and the systems (1.2) and (1.3) are uniformly persistent and admit at least one EE provided that $\mathcal{R}_0 > 1$, respectively.

One of the highlights of this work is that, under certain conditions, we seem to derive for the first time the variational expression of $\mathcal{R}_0$ for vector-borne disease models by means of the variational method [32]. With the aid of the variational formula, we examined the asymptotic behaviors of $\mathcal{R}_0$ w.r.t the diffusion and advection rates (Proposition 4.2). In particular, we found that when the diffusion rates (D_2, D_5) go arbitrarily small or the advection rates (q_h, q_v) go arbitrarily large, even if the downstream end is a high-risk site, the disease will eventually be eradicated. Biologically, this is mainly due to the fact that the downstream environment is not conducive to the survival of hosts and vectors, which is in contrast to Theorem 3.1 in [28]. Furthermore, we discussed the monotonicity of $\mathcal{R}_0$ w.r.t q_h and q_v. When $\nu_i \in [1/2, \infty]$, $i = 1, 2$, $\mathcal{R}_0$ is a monotone decreasing function of q_h and q_v, respectively. However, the monotonicity is not necessarily valid (see [28, Remark 4.3]) when the boundary conditions are selected as no-flux type (i.e., $\nu_1 = \nu_2 = 0$). Then, we classified the dynamics of (1.2), and obtained interesting and important phenomena (Theorems 5.1 and 5.2). When the habitat locates in a high-risk area and $\mathcal{R}_{02}^{\mathrm{a}} > 1$, there are unique surfaces $\chi_1(D_2, D_5)$ and $\chi_2(D_2, D_5)$, such that DFE is GAS as $q_h > \chi_1(D_2, D_5)$ or $q_v > \chi_2(D_2, D_5)$, and system (1.2) is uniformly persistent and has at least one EE as $0 < q_h < \chi_1(D_2, D_5)$ or $0 < q_v < \chi_2(D_2, D_5)$. When the habitat locates in a low-risk area and $k_1(x, P_0)k_2(x, A_0) - \delta_h(x)d_v(x)$ changes sign in $(0, L)$, there are unique critical points D_2^* and D_5^*, such that when $(D_2, D_5) \in (0, D_2^*) \times (0, D_5^*)$, there are unique surfaces $\chi_3(D_2, D_5)$ and $\chi_4(D_2, D_5)$ such that DFE is GAS as $q_h > \chi_3(D_2, D_5)$ or $q_v > \chi_4(D_2, D_5)$, and system (1.2) is uniformly persistent and has at least one EE as $0 < q_h < \chi_3(D_2, D_5)$ or $0 < q_v < \chi_4(D_2, D_5)$; When $(D_2, D_5) \in (D_2^*, \infty) \times (D_5^*, \infty)$, DFE is always GAS for any $q_h, q_v > 0$. It should be pointed out that $k_1(\cdot, P) = k_2(\cdot, A)$ is mathematically a technical condition that makes $\mathcal{F}(\cdot)$ a symmetric matrix. In addition, since the asymptotic profile of $\mathcal{R}_0$ for system (1.3) with Dirichlet type boundary conditions remains unclear in Proposition 4.2 (iii), the dynamic classification of (1.3) is not conclusive, which is reserved for future investigation.

This paper supplements and promotes the relevant results in [28]. In some circumstances, the downstream environment has different effects on the survival of hosts and vectors (mathematically, the downstream corresponds to hybrid boundary conditions). Therefore, it is interesting and necessary to study the influence of advection effect on vector-borne disease transmission under hybrid boundary conditions. On the other hand, one simplification of our model is constant diffusion and advection coefficients. Nevertheless, the movement strategies of hosts and vectors depend on the spatial heterogeneity of the habitat and should follow the general model derivations in [21,22]. We will also explore the spatial dynamics of models (1.2) and (1.3) under the heterogeneous diffusion and advection.

Data availability

No data was used for the research described in the article.

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