Solutions to Assignment #1

EX 1.4.1. $$c\rho \frac{\partial u}{\partial t} = K_0 \frac{\partial^2 u}{\partial x^2} + Q$$

Equilibrium satisfies $\quad K_0 \frac{d^2 u}{dx^2} + Q = 0 \quad$ (u is independent of t)

(a)
$$\frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = c_1 x + c_2$$
$$u(0) = 0 \Rightarrow c_2 = 0$$
$$u(L) = T \Rightarrow c_1 L = T \Rightarrow c_1 = \frac{T}{L}$$
Equilibrium: $\boxed{u(x) = \frac{T}{L}x}$

(b)
$$\frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = c_1 x + c_2$$
$$u(0) = T \Rightarrow c_2 = T$$
$$u(L) = 0 \Rightarrow c_1 L + T = 0 \Rightarrow c_1 = -\frac{T}{L}$$
Equilibrium: $\boxed{u(x) = -\frac{T}{L}x + T}$

(c)
$$\frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = c_1 x + c_2$$
$$\frac{du}{dx} = c_1$$
$$\left.\begin{aligned} \frac{du}{dx}(0) = 0 &\Rightarrow c_1 = 0 \\ u(L) = T &\Rightarrow c_2 = T \end{aligned}\right\} \Rightarrow \text{Equilibrium: } \boxed{u(x) = T}$$

(d)
$$\frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = c_1 x + c_2$$
$$\frac{du}{dx} = c_1$$
$$\left.\begin{aligned} u(0) = T &\Rightarrow c_2 = T \\ \frac{du}{dx}(L) = \alpha &\Rightarrow c_1 = \alpha \end{aligned}\right\} \Rightarrow \text{Equilibrium: } \boxed{u(x) = \alpha x + T}$$

(e)
$$K_0 \frac{d^2u}{dx^2} + K_0 = 0 \Rightarrow \frac{d^2u}{dx^2} = -1 \Rightarrow \frac{du}{dx} = -x + c_1$$
$$\Rightarrow u(x) = -\frac{1}{2}x^2 + c_1 x + c_2$$
$$\left.\begin{aligned} u(0) = T_1 &\Rightarrow c_2 = T_1 \\ u(L) = T_2 &\Rightarrow -\frac{1}{2}L^2 + c_1 L + T_1 = T_2 \Rightarrow c_1 = \frac{\frac{1}{2}L^2 + T_2 - T_1}{L} \end{aligned}\right\}$$
Equilibrium: $\boxed{u(x) = -\frac{1}{2}x^2 + \frac{\frac{1}{2}L^2 + T_2 - T_1}{L}x + T_1}$

(f) $K_0 \frac{d^2u}{dx^2} + K_0 x^2 = 0 \Rightarrow \frac{d^2u}{dx^2} = -x^2 \Rightarrow \frac{du}{dx} = -\frac{1}{3}x^3 + C_1$

$\Rightarrow u(x) = -\frac{1}{12}x^4 + C_1 x + C_2$

$u(0) = T \Rightarrow C_2 = T$

$\frac{du}{dx}(L) = 0 \Rightarrow -\frac{1}{3}L^3 + C_1 = 0 \Rightarrow C_1 = \frac{1}{3}L^3$

Equilibrium: $\boxed{u(x) = -\frac{1}{12}x^4 + \frac{1}{3}L^3 x + T}$

(g) $\frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = C_1 x + C_2$

$\frac{du}{dx} = C_1$

$u(0) = T \Rightarrow C_2 = T$

$\frac{du}{dx}(L) + u(L) = 0 \Rightarrow C_1 + C_1 L + T = 0 \Rightarrow C_1 = -\frac{T}{1+L}$

Equilibrium: $\boxed{u(x) = -\frac{T}{1+L}x + T}$

(h) $\frac{d^2u}{dx^2} = 0 \Rightarrow u(x) = C_1 x + C_2$

$\frac{du}{dx} = C_1$

$\frac{du}{dx}(0) - [u(0) - T] = 0 \Rightarrow C_1 - [C_2 - T] = 0$

$\Rightarrow C_1 - C_2 + T = 0$

$\frac{du}{dx}(L) = \alpha \Rightarrow C_1 = \alpha$

$\Rightarrow C_2 = \alpha + T$

Equilibrium: $\boxed{u(x) = \alpha x + \alpha + T}$

Note: in this problem (Ex 1.4.1), all boundary conditions should be expressed by derivatives instead of partial derivatives!

EX 12.2.2.

$$\frac{dw}{dt} = 0 \quad \text{if} \quad \frac{dx}{dt} = -3$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\boxed{W(x,t)} = W(x_0, 0) \qquad x = -3t + x_0$$

$$= \cos x_0 \qquad \Downarrow$$

$$x_0 = x + 3t$$

$$\boxed{= \cos(x+3t)}$$

EX 12.2.5.

(a)

$$\frac{dw}{dt} = e^{2x} \quad \text{if} \quad \frac{dx}{dt} = c$$

$$\Downarrow$$

$$\frac{dw}{dt} = e^{2(ct+x_0)} \qquad x = ct + x_0$$

$$= e^{2ct+2x_0} \qquad \Downarrow$$

$$x_0 = x - ct$$

$$\cancel{\boxed{W} = W(x_0,0)e^{2ct} = f(x_0)e^{2ct} \boxed{= f(x-ct)e^{2ct}}}$$

$$W = \frac{1}{2c}e^{2x_0}e^{2ct} + \alpha$$

When $t=0$, $W = W(x_0, 0)$.

So $W(x_0,0) = \frac{1}{2c}e^{2x_0} + \alpha$

$$\Rightarrow \alpha = W(x_0,0) - \frac{e^{2x_0}}{2c}$$

Hence, $\boxed{W(x,t)} =$

$$\frac{e^{2x_0}}{2c}e^{2ct} + W(x_0,0) - \frac{e^{2x_0}}{2c}$$

$$= \frac{e^{2(x-ct)}}{2c}e^{2ct} + f(x-ct) - \frac{e^{2(x-ct)}}{2c}$$

$$\boxed{= \frac{e^{2x}}{2c} - \frac{e^{2x-2ct}}{2c} + f(x-ct)}$$

(b)

$$\frac{dw}{dt} = 1 \quad \text{if} \quad \frac{dx}{dt} = x$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\boxed{W(x,t)} = W(x_0,0) + t \qquad x = x_0 e^{t}$$

$$= f(x_0) + t \qquad \Downarrow$$

$$x_0 = xe^{-t}$$

$$\boxed{= f(xe^{-t}) + t}$$

(c)

$$\frac{dw}{dt} = 1 \quad \text{if} \quad \frac{dx}{dt} = t$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\boxed{W(x,t)} = W(x_0,0) + t \qquad x = \frac{1}{2}t^2 + x_0$$

$$= f(x_0) + t \qquad \Downarrow$$

$$x_0 = x - \frac{1}{2}t^2$$

$$\boxed{= f(x - \tfrac{1}{2}t^2) + t}$$

(d) $\dfrac{dw}{dt} = w$ if $\dfrac{dx}{dt} = 3t$

$\Downarrow$ $\Downarrow$

$\boxed{w(x,t)} = w(x_0, 0)e^t$ $x = \frac{3}{2}t^2 + x_0$

$= f(x_0)e^t$ $\Downarrow$

$x_0 = x - \frac{3}{2}t^2$

$$\boxed{= f(x - \tfrac{3}{2}t^2)e^t}$$

EX 12.3.4

(a) $\dfrac{\partial u}{\partial t} = F'(x-ct)(-c)$

$\boxed{\dfrac{\partial u}{\partial t}(x,0) = -cF'(x)}$ (same as $-c\dfrac{dF(x)}{dx}$)

(b) $\dfrac{\partial u}{\partial x} = F'(x-ct)\cdot 1$

$\boxed{\dfrac{\partial u}{\partial x}(0,t) = F'(-ct)}$ (same as $\dfrac{dF(-ct)}{d(-ct)} = -\dfrac{1}{c}\cdot\dfrac{dF(-ct)}{dt}$)

EX 12.3.5

d'Alembert's solution: $u(x,t) = \dfrac{f(x-ct) + f(x+ct)}{2} + \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$

$= \dfrac{1}{2c}\displaystyle\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x}$

$(f(x) = 0)$

Six cases or ten cases ($t < h/c$, $t > h/c$)

1. $x-ct < x+ct < -h$, $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 0\,d\bar{x} = 0$
2. $x-ct < -h < x+ct < h$, $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} g(\bar{x})\,d\bar{x} = \frac{1}{2c}\left(\int_{x-ct}^{-h} 0\,d\bar{x} + \int_{-h}^{x+ct} 1\,d\bar{x}\right) = \frac{x+ct+h}{2c}$
3. $x-ct < -h < h < x+ct$, $u(x,t) = \frac{1}{2c}\left(\int_{x-ct}^{-h} + \int_{-h}^{h} + \int_{h}^{x+ct}\right) g(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_{-h}^{h} 1\,d\bar{x} = \frac{h}{c}$
4. $-h < x-ct < x+ct < h$, $u(x,t) = \frac{1}{2c}\int_{x-ct}^{x+ct} 1\,d\bar{x} = t$
5. $-h < x-ct < h < x+ct$, $u(x,t) = \frac{1}{2c}\left(\int_{x-ct}^{h} + \int_{h}^{x+ct}\right) g(\bar{x})\,d\bar{x} = \frac{1}{2c}\int_{x-ct}^{h} 1\,d\bar{x} = \frac{h-x+ct}{2c}$
6. $h < x-ct < x+ct$, $u(x,t) = \int_{x-ct}^{x+ct} 0\,d\bar{x} = 0$

For ten cases as Hint suggests,

$t < h/c$, five cases 1 2 4 5 6, results are in previous page;

$t > h/c$, five cases 1 2 3 5 6, results are in previous page.

EX 12.6.7.

(a) $\frac{d\rho}{dt} = 0$ if $\frac{dx}{dt} = \rho^2$

$\rho = \rho(x_0, 0)$ $\quad = \rho(x_0,0)^2$

$\Downarrow$

$x = \rho(x_0,0)^2 t + x_0 \quad (*)$

When $x_0 < 0$, $\rho = \rho(x_0, 0) = 3$

$x = 3^2 t + x_0 = 9t + x_0 < 9t$

When $x_0 > 0$, $\rho = \rho(x_0,0) = 4$

$x = 4^2 t + x_0 = 16t + x_0 > 16t$

when $x_0 = 0$, $x = \rho^2 t$ from $(*)$, $3 < \rho < 4$

$\Rightarrow$ $\rho = \sqrt{\frac{x}{t}}$ and $9t < x < 16t$

Hence,

$$\boxed{\rho_{(x,t)} = \begin{cases} 3, & x < 9t \\ \sqrt{x/t}, & 9t < x < 16t \\ 4, & x > 16t \end{cases}}$$

(b) $\frac{d\rho}{dt} = 0$, if $\frac{dx}{dt} = 4\rho$

$\Downarrow$

$\rho = \rho(x_0, 0)$ $\quad = 4\rho(x_0,0) \Rightarrow x = 4\rho(x_0,0)t + x_0 \quad (**)$

when $x_0 < 1$, $\rho = \rho(x_0, 0) = 2$, $x = 8t + x_0 < 8t + 1$

When $x_0 > 1$, $\rho = \rho(x_0,0) = 3$, $x = 12t + x_0 > 12t + 1$

When $x_0 = 1$, $x = 4\rho t + 1$ from $(**)$, $2 < \rho < 3$

$$\Rightarrow \rho = \frac{x-1}{4t} \quad \text{and} \quad 8t+1 < x < 12t+1$$

Hence,

$$\boxed{\rho(x,t) = \begin{cases} 2, & x < 8t+1 \\ \frac{x-1}{4t}, & 8t+1 < x < 12t+1 \\ 3, & x > 12t+1 \end{cases}}$$

(c)

$$\frac{d\rho}{dt} = 0 \quad \text{if} \quad \frac{dx}{dt} = 3\rho$$

$$\Downarrow$$

$$\rho = \rho(x_0, 0) \qquad = 3\rho(x_0,0) \quad \Rightarrow \quad x = 3\rho(x_0,0)t + x_0 \quad (***)$$

When $x_0 < 0$, $\rho = \rho(x_0, 0) = 1$, $x = 3t + x_0 < 3t$

When $0 < x_0 < 1$, $\rho = \rho(x_0, 0) = 2$, $x = 6t + x_0 \in (6t, 6t+1)$

When $x_0 > 1$, $\rho = \rho(x_0, 0) = 4$, $x = 12t + x_0 > 12t+1$

When $x_0 = 0$, $x = 3\rho t$ from $(***)$, $1 < \rho < 2$

$$\Rightarrow \rho = \frac{x}{3t} \quad \text{and} \quad 3t < x < 6t$$

When $x_0 = 1$, $x = 3\rho t + 1$ from $(***)$, $2 < \rho < 4$

$$\Rightarrow \quad \rho = \frac{x-1}{3t} \quad \text{and} \quad 6t+1 < x < 12t+1$$

Hence,

$$\boxed{\rho(x,t) = \begin{cases} 1, & x < 3t \\ x/3t, & 3t < x < 6t \\ 2, & 6t < x < 6t+1 \\ \frac{x-1}{3t}, & 6t+1 < x < 12t+1 \\ 4, & x > 12t+1 \end{cases}}$$

Ex 12.6.8

(a) $\frac{d\rho}{dt} = e^{-3x}$ if $\frac{dx}{dt} = c$

$\Downarrow$

$= e^{-3(ct+x_0)}$ $\qquad x = ct + x_0 \Rightarrow x_0 = x - ct$

$$\Rightarrow \rho = \frac{e^{-3x_0}}{-3c} e^{-3ct} + \alpha$$

when $t=0$, $\rho = \rho(x_0, 0) \Rightarrow \rho(x_0,0) = \frac{e^{-3x_0}}{-3c} + \alpha$

$$\Rightarrow \alpha = \rho(x_0,0) + \frac{e^{-3x_0}}{3c}$$

Hence, $\boxed{\rho(x,t)} = \frac{e^{-3x_0}}{-3c} e^{-3ct} + \rho(x_0,0) + \frac{e^{-3x_0}}{3c}$

$$= \frac{e^{-3(x-ct)}}{-3c} e^{-3ct} + f(x-ct) + \frac{e^{-3(x-ct)}}{3c}$$

$$\boxed{= \frac{e^{-3x}}{-3c} + \frac{e^{-3x+3ct}}{3c} + f(x-ct)}$$

(same as the book's solution)

(b) $\frac{d\rho}{dt} = 4$ if $\frac{dx}{dt} = 3x$

$\Downarrow$ $\qquad \Downarrow$

$x = x_0 e^{3t}$

$\Downarrow$

$\boxed{\rho} = 4t + \rho(x_0, 0)$ $\qquad x_0 = xe^{-3t}$

$= 4t + f(x_0)$

$\boxed{= 4t + f(xe^{-3t})}$

(c) $\frac{d\rho}{dt} = 5$ if $\frac{dx}{dt} = t$

$\Downarrow$ $\qquad \Downarrow$

$\boxed{\rho} = 5t + \rho(x_0, 0)$ $\qquad x = \frac{1}{2}t^2 + x_0$

$\Downarrow$

$= 5t + f(x_0)$ $\qquad x_0 = x - \frac{1}{2}t^2$

$\boxed{= 5t + f(x - \frac{1}{2}t^2)}$

(d) $\frac{d\rho}{dt} = 3\rho$ if $\frac{dx}{dt} = 5t$

$\Downarrow$ $\Downarrow$

$\boxed{\rho} = \rho(x_0, 0)e^{3t}$ $x = \frac{5}{2}t^2 + x_0$

$= f(x_0)e^{3t}$ $\Downarrow$

$x_0 = x - \frac{5}{2}t^2$

$\boxed{= f(x - \frac{5}{2}t^2)e^{3t}}$

(e) $\frac{d\rho}{dt} = -\rho$ if $\frac{dx}{dt} = -t^2$

$\Downarrow$ $\Downarrow$

$\boxed{\rho} = \rho(x_0, 0)e^{-t}$ $x = -\frac{1}{3}t^3 + x_0$

$= f(x_0)e^{-t}$ $\Downarrow$ $x_0 = x + \frac{1}{3}t^3$

$\boxed{= f(x + \frac{1}{3}t^3)e^{-t}}$

(f) $\frac{d\rho}{dt} = 0$ if $\frac{dx}{dt} = t^2$

$\Downarrow$ $\Downarrow$

$\boxed{\rho} = \rho(x_0, 0)$ $x = \frac{1}{3}t^3 + x_0$

$\Downarrow$

$= f(x_0)$ $x_0 = x - \frac{1}{3}t^3$

$\boxed{= f(x - \frac{1}{3}t^3)}$

(g) $\frac{d\rho}{dt} = t$ if $\frac{dx}{dt} = x$

$\Downarrow$ $\Downarrow$

$\boxed{\rho} = \frac{1}{2}t^2 + \rho(x_0, 0)$ $x = x_0 e^t$

$\Downarrow$

$= \frac{1}{2}t^2 + f(x_0)$ $x_0 = xe^{-t}$

$\boxed{= \frac{1}{2}t^2 + f(xe^{-t})}$

Ex 12.6.1

(a) $\quad \dfrac{d\rho}{dt} = 3\rho$ if $\dfrac{dx}{dt} = -\rho^2$

$\Downarrow$

$\rho = \rho(x_0,0)e^{3t}$ $\qquad = -\rho(x_0,0)^2 e^{6t}$

$\Downarrow$

$$x = -\frac{\rho(x_0,0)^2}{6} e^{6t} + \alpha$$

when $t=0$, $x = x_0$

$$\Rightarrow \quad x_0 = -\frac{\rho(x_0,0)^2}{6} + \alpha$$

$$\Rightarrow \quad \alpha = x_0 + \frac{\rho(x_0,0)^2}{6}$$

Hence

$$x = -\frac{\rho(x_0,0)^2}{6} e^{6t} + x_0 + \frac{\rho(x_0,0)^2}{6}$$

A parametric representation of the solution:

$$\begin{cases} \rho(x,t) = f(x_0)e^{3t} \\ x = -\dfrac{f(x_0)^2}{6} e^{6t} + x_0 + \dfrac{f(x_0)^2}{6} \end{cases}$$

i.e.

$$\boxed{\begin{cases} \rho(x,t) = f(x_0)e^{3t} \\ x = \dfrac{1}{6}(1-e^{6t}) f^2(x_0) + x_0 \end{cases}}$$

(Note that $f(x_0)^2$ and $f^2(x_0)$ are the same thing.)

(b) $\quad \dfrac{d\rho}{dt} = t$ if $\dfrac{dx}{dt} = \rho$

$\Downarrow$

$\rho = \frac{1}{2}t^2 + \rho(x_0,0)$ $\qquad = \frac{1}{2}t^2 + \rho(x_0,0)$

$\Downarrow$

$$x = \frac{1}{6}t^3 + \rho(x_0,0)\,t + x_0$$

Hence,

$$\boxed{\begin{cases} \rho(x,t) = \frac{1}{2}t^2 + f(x_0) \\ x = \frac{1}{6}t^3 + f(x_0)\,t + x_0 \end{cases}}$$

(c) $\frac{d\rho}{dt} = -\rho$ 且 $\frac{dx}{dt} = t^2\rho$

$\Downarrow$

$\rho = \rho(x_0,0)e^{-t}$ $\qquad = t^2\rho(x_0,0)\,e^{-t}$

$\Downarrow$ (Integration by parts TWICE!)

$x = \rho(x_0,0)e^{-t}(-t^2-2t-2) + \alpha$

when $t=0$, $x = x_0$

$\Rightarrow x_0 = \rho(x_0,0)(-2) + \alpha$

$\Rightarrow \alpha = x_0 + 2\rho(x_0,0)$

Hence, $x = \rho(x_0,0)(-t^2-2t-2)e^{-t} + x_0 + 2\rho(x_0,0)$

Therefore,

$$\boxed{\begin{cases} \rho(x,t) = f(x_0)e^{-t} \\ x = x_0 + f(x_0)\left[(-t^2-2t-2)e^{-t} + 2\right] \end{cases}}$$

(Note that $\int_0^t \tau^2 e^{-\tau}d\tau = (-t^2-2t-2)e^{-t} + 2$)