Solution to Assignment #5

Ex 8.3.1 (a) The related homogeneous problem

$$\begin{cases} \frac{\partial u}{\partial t} = k\frac{\partial^2 u}{\partial x^2} \\ u(0,t) = 0 \\ \frac{\partial u}{\partial x}(L,t) = 0 \end{cases}$$

has eigenvalues $\lambda_n = \left(\frac{(2n-1)\pi}{2L}\right)^2$, $n = 1, 2, 3, \cdots$

eigenfunctions $\phi_n(x) = \sin\frac{(2n-1)\pi x}{2L}$.

Eigenfunction expansion of the nonhomogeneous problem:

$$u(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)$$

Substitute into the nonhomogeneous PDE to obtain

$$\sum_{n=1}^{\infty}\left[\frac{da_n}{dt} + \lambda_n k a_n\right]\phi_n(x) = Q(x,t)$$

$$\frac{da_n}{dt} + \lambda_n k a_n = \frac{\int_0^L Q(x,t)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx} \triangleq q(t)$$

Use Integrating factor $e^{\lambda_n k t}$ to obtain

$$a_n(t) = a_n(0)e^{-\lambda_n k t} + e^{-\lambda_n k t}\int_0^t q_n(\tau)e^{\lambda_n k \tau}\,d\tau$$

I.C.: $u(x,0) = f(x) \Rightarrow \sum_{n=1}^{\infty} a_n(0)\phi_n(x) = f(x)$

$$\Rightarrow a_n(0) = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2(x)\,dx}.$$

Ex 8.3.6. Choose the reference function $r(x,t) = 1 - \frac{x}{\pi}$.

Let $v(x,t) = u(x,t) - r(x,t)$, then

$$\begin{cases} \frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2} + \sin 5x\, e^{-2t} \\ v(0,t) = 0 \\ v(\pi,t) = 0 \\ v(x,0) = -1 + \frac{x}{\pi}. \end{cases}$$

The related homogeneous problem has

eigenvalues $\lambda_n = n^2$, $n = 1, 2, 3, \cdots$

eigenfunctions $\phi_n(x) = \sin nx$.

Eigenfunction expansion $v(x,t) = \sum_{n=1}^{\infty} a_n(t)\,\phi_n(x)$

Substitute into the nonhomogeneous PDE to obtain

$$\sum_{n=1}^{\infty} \left[\frac{da_n}{dt} + \lambda_n a_n\right] \phi_n(x) = \sin 5x\, e^{-2t}$$

$$\frac{da_n}{dt} + n^2 a_n = \frac{2}{\pi}\int_0^{\pi} \sin nx \sin 5x\, e^{-2t}\, dx$$

$$= \frac{2e^{-2t}}{\pi}\int_0^{\pi} \sin nx \sin 5x\, dx$$

$$= \begin{cases} e^{-2t}, & n = 5 \\ 0, & n \neq 5 \end{cases}$$

Use integrating factor $e^{n^2 t}$ to obtain

$$a_n(t) = \begin{cases} \left(a_5(0) - \frac{1}{23}\right) e^{-25t} + \frac{1}{23} e^{-2t}, & n = 5. \\ a_n(0)\, e^{-n^2 t}, & n \neq 5 \end{cases}$$

I.C.: $v(x,0) = -1 + \frac{x}{\pi} \Rightarrow \sum_{n=1}^{\infty} a_n(0)\,\phi_n(x) = -1 + \frac{x}{\pi}$

$$\Rightarrow a_n(0) = \frac{2}{\pi}\int_0^{\pi} \left(-1 + \frac{x}{\pi}\right) \sin nx\, dx$$

$$= -\frac{2}{n\pi}$$

The solution to the original nonhomogeneous problem is

$u(x,t) = v(x,t) + r(x,t) = \cdots\cdots$

Ex 8.3.7. Choose a reference function

$$r(x,t) = \frac{xt}{L}$$

(Note: you can also choose the reference function $r(x,t) = \frac{x^3}{6L} + \left(\frac{t}{L} - \frac{L}{6}\right)x$ proposed in textbook.)

Let $v(x,t) = u(x,t) - r(x,t)$, then

$$\begin{cases} \dfrac{\partial v}{\partial t} = \dfrac{\partial^2 v}{\partial x^2} - \dfrac{x}{L} \\ v(0,t) = 0 \\ v(L,t) = 0 \\ v(x,0) = 0 \end{cases}$$

The associated homogeneous problem has

eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n = 1, 2, 3, \ldots$

eigenfunctions $\phi_n(x) = \sin\frac{n\pi x}{L}$.

Eigenfunction Expansion $v(x,t) = \sum_{n=1}^{\infty} a_n(t)\phi_n(x)$

substitute into the nonhomogeneous PDE:

$$\sum_{n=1}^{\infty}\left[\frac{da_n}{dt} + \lambda_n a_n\right]\phi_n(x) = -\frac{x}{L}$$

$$\Rightarrow \frac{da_n}{dt} + \lambda_n a_n = \frac{2}{L}\int_0^L \left(-\frac{x}{L}\right)\sin\frac{n\pi x}{L}\,dx = \frac{2\cdot(-1)^n}{n\pi}$$

$$\Rightarrow a_n(t) = a_n(0)e^{-\lambda_n t} + e^{-\lambda_n t}\int_0^t \frac{2(-1)^n}{n\pi}e^{\lambda_n \tau}\,d\tau$$

$$= a_n(0)e^{-\lambda_n t} + e^{-\lambda_n t}\cdot\frac{2(-1)^n}{n\pi\lambda_n}\left(e^{\lambda_n t} - 1\right)$$

$$= \left[a_n(0) - \frac{2(-1)^n}{n\pi\lambda_n}\right]e^{-\lambda_n t} + \frac{2(-1)^n}{n\pi\lambda_n}.$$

I.C.: $v(x,0)=0 \Rightarrow \sum_{n=1}^{\infty} a_n(0)\,\phi_n(x)=0$

$$\Rightarrow \quad a_n(0)=0$$

Hence $\quad a_n(t) = \left(1-e^{-\lambda_n t}\right)\dfrac{2(-1)^n}{n\pi\lambda_n}$.

The solution to the original nonhomogeneous problem is $\quad u(x,t) = v(x,t) + r(x,t) = \cdots\cdots$.

Ex 8.4.1 (b). The associated homogeneous problem has eigenvalues $\lambda_n = \left(\frac{n\pi}{L}\right)^2$, $n=0,1,2,3,\cdots$

eigenfunctions $\phi_n(x) = \cos\frac{n\pi x}{L}$.

Eigenfunction Expansion $\quad u(x,t)=\sum_{n=0}^{\infty} b_n(t)\,\phi_n(x)$, then

$$\frac{db_n}{dt} = q_n(t) + \frac{k\int_0^L \frac{\partial^2 u}{\partial x^2}\phi_n(x)\,dx}{\int_0^L \phi_n^2\,dx},$$

where

$$q_n(t) = \frac{\int_0^L Q(x,t)\,\phi_n(x)\,dx}{\int_0^L \phi_n^2\,dx}.$$

Green's formula: $\displaystyle\int_0^L \left(u\frac{\partial^2\phi_n}{\partial x^2} - \phi_n\frac{\partial^2 u}{\partial x^2}\right)dx = \left(u\frac{\partial\phi_n}{\partial x} - \phi_n\frac{\partial u}{\partial x}\right)\Big|_0^L$

$$\Rightarrow -\lambda_n\int_0^L u\phi_n\,dx - \int_0^L \phi_n\frac{\partial^2 u}{\partial x^2}\,dx = -\phi_n(L)\frac{\partial u}{\partial x}(L,t) + \phi_n(0)\frac{\partial u}{\partial x}(0,t)$$

$$\Rightarrow \int_0^L \phi_n\frac{\partial^2 u}{\partial x^2}\,dx = -\lambda_n\int_0^L u\phi_n\,dx + \boxed{(-1)^n B(t) - A(t)}$$

$$\frac{db_n}{dt} = q_n(t) + \frac{-k\lambda_n \int_0^L u\phi_n dx + k\left[(-1)^n B(t) - A(t)\right]}{\int_0^L \phi_n^2 dx}$$

$$= q_n(t) - k\lambda_n b_n + \frac{k\left[(-1)^n B(t) - A(t)\right]}{\int_0^L \phi_n^2 dx}$$

$$\Rightarrow \quad \frac{db_n}{dt} + k\lambda_n b_n = q_n(t) + \frac{k\left[(-1)^n B(t) - A(t)\right]}{\int_0^L \phi_n^2 dx}$$

Use integrating factor $e^{k\lambda_n t}$ to solve this first-order ODE, and determine $b_n(0)$ from I.C.

I.C.: $u(x,0) = f(x) \Rightarrow \sum_{n=0}^{\infty} b_n(0)\phi_n(x) = f(x)$

$$\Rightarrow b_n(0) = \frac{\int_0^L f(x)\phi_n(x)\,dx}{\int_0^L \phi_n^2 dx}, \quad n = 0, 1, 2, \dots$$

(formulas for Fourier cosine coefficients).

Ex 10.3.1. (a) $\mathcal{F}[c_1 f(x) + c_2 g(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}[c_1 f(\bar{x}) + c_2 g(\bar{x})]e^{i\omega\bar{x}}d\bar{x}$

$$= \frac{c_1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})e^{i\omega\bar{x}}d\bar{x} + \frac{c_2}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})e^{i\omega\bar{x}}d\bar{x} = c_1\mathcal{F}[f(x)] + c_2\mathcal{F}[g(x)]$$

(b) $\mathcal{F}[f(x)g(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})g(\bar{x})e^{i\omega\bar{x}}d\bar{x}$ $\times$

$$F(\omega)G(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\bar{x})e^{i\omega\bar{x}}d\bar{x}\cdot\frac{1}{2\pi}\int_{-\infty}^{\infty} g(\bar{x})e^{i\omega\bar{x}}d\bar{x}$$

Ex 10.3.2.

(a) $\mathcal{F}^{-1}[c_1F(w)+c_2G(w)] = \int_{-\infty}^{\infty}[c_1F(w)+c_2G(w)]e^{-iwx}dw$

$= c_1\int_{-\infty}^{\infty}F(w)e^{-iwx}dw + c_2\int_{-\infty}^{\infty}G(w)e^{-iwx}dw$

$= c_1 f(x) + c_2 g(x)$

(b) $\mathcal{F}^{-1}[F(w)G(w)] = \int_{-\infty}^{\infty}F(w)G(w)e^{-iwx}dw$

$f(x)g(x) = \int_{-\infty}^{\infty}F(w)e^{-iwx}dw \cdot \int_{-\infty}^{\infty}G(w)e^{-iwx}dw$

Ex 10.3.5. $\mathcal{F}^{-1}[e^{iw\beta}F(w)] = \int_{-\infty}^{\infty}e^{iw\beta}F(w)e^{-iwx}dw$

$= \int_{-\infty}^{\infty}F(w)e^{-iw(x-\beta)}dw = f(x-\beta)$.

Ex 10.3.6. $\mathcal{F}[f(x)] = \frac{1}{2\pi}\int_{-\infty}^{\infty}f(\bar{x})e^{iw\bar{x}}d\bar{x}$

$= \frac{1}{2\pi}\int_{-a}^{a}e^{iw\bar{x}}d\bar{x} = \frac{1}{\pi}\cdot\frac{\sin aw}{w}$

Ex 10.3.7. $\mathcal{F}^{-1}[e^{-|w|\alpha}] = \int_{-\infty}^{\infty}e^{-|w|\alpha}e^{-iwx}dw$

$= \int_{0}^{\infty}e^{-w\alpha}e^{-iwx}dw + \int_{-\infty}^{0}e^{w\alpha}e^{-iwx}dw$

$= \int_{0}^{\infty}e^{-(\alpha+ix)w}dw + \int_{-\infty}^{0}e^{(\alpha-ix)w}dw$

$$= \frac{1}{-(\alpha+ix)} e^{-(\alpha+ix)w}\Big|_0^{\infty} + \frac{1}{\alpha-ix} e^{(\alpha-ix)w}\Big|_{-\infty}^{0}$$

$$= \frac{1}{\alpha+ix} + \frac{1}{\alpha-ix} = \frac{2\alpha}{\alpha^2+x^2}.$$

Ex 10.4.10. See Section 10.6.1 of textbook.

* After find $U(w,t)$, we use definition of inverse Fourier transform directly to obtain $u(x,t)$.

Here, convolution theorem can't work!